A linear path towards self-synchronization: Analysis of the fully locked transition of the Kuramoto model

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***Abstract*— We present a linear reformulation of the Kuramoto model describing a self-synchronizing phase transition in a heterogeneous system of globally coupled oscillators that in general have different characteristic frequencies. While this approach can also be applied to systems with a finite number of oscillators, discussion here will focus on the reformulated model in the continuum limit, the regime of validity of the original Kuramoto solution. This new approach allows one to solve explicitly for the synchronization order parameter and the critical point for a new class of continuum systems that have no solution through the traditional approach to the Kuramoto model. Furthermore, the synchronization order parameter will be shown to exhibit anomalous scaling in the vicinity of the critical point. This novel linear approach appears to be a promising way to extend the applicability of the Kuramoto model, which is the paradigm of spontaneous synchronization. Although discussion here will be restricted to systems with global coupling, the formalism of the linear approach also lends itself to solving systems that exhibit local or asymmetric coupling.**

I. Introduction

The Kuramoto model of self-synchronizing coupled phase oscillators is recognized as important for being able to describe diverse synchronization phenomena such as collective atomic recoil lasing, the behavior of Josephson junction arrays, and neural firing patterns [1][2][3]. In a broader sense, however, the Kuramoto model is an exactly solvable model that exhibits behavior reminiscent of a nonequilibrium phase transition. As such, it is a useful medium through which we can develop a better comprehension of nonequilibrium systems and, to this end, in this paper we seek to extend the Kuramoto solution. Certainly, a more general Kuramoto solution will also broaden the applicability of the Kuramoto model as a paradigm of spontaneous synchronization.

To generalize the Kuramoto solution we will take a linear approach, which will phrase this nonlinear problem in terms of eigenvalues and eigenvectors, opening it up to spectral theory and other tools and powerful techniques developed for solving linear problems. Note that only the fully locked transition will be considered here.

We shall begin by presenting a linear model that maps onto the Kuramoto model and deriving the general solution it yields for the synchronization order parameter and the critical point of a continuum system of oscillators. We then focus on a system with a particular coupling scheme, which cannot be solved using the traditional approach. We present the solution for this case, demonstrate that the synchronization order parameter has anomalous scaling about the critical point, and apply this linear approach to oscillator systems with different characteristic frequency distributions.

While the discussion here will cover only the continuum limit and global coupling, which is the regime of validity of the traditional Kuramoto solution, the linear approach presented can also be used to solve systems populated by a finite number of oscillators [5] as well as systems with local or asymmetric coupling. For further details of these results, see [5] and [6].

II. Linear reformulation for generalized coupling

For a system of coupled oscillators in the continuum limit, the linear reformulation of the Kuramoto model to be discussed here can be expressed as

$$\dot{\psi}(\omega,t) = (i\omega-\gamma)\psi(\omega,t) + \int_{-\infty}^{\infty}\Omega(\omega,\omega')g(\omega')\psi(\omega',t)d\omega', \tag{1}$$

where $\Omega(\omega,\omega')$ describes the coupling between pairs of oscillators with characteristic frequencies ω and ω' respectively, the phases of the complex variable $\psi(\omega,t)$ correspond to those of the system's oscillators whose synchronization properties we investigate, and $g(\omega)$ is the distribution of their characteristic frequencies. γ is a parameter fixed according to the system parameters such that the amplitude of $\psi(\omega,t)$ goes to a steady state in the long-time limit. This allows the linear model to be mapped onto the original Kuramoto model in the synchronized region, since with the nonlinear transformation $\psi(\omega) = R(\omega)e^{i\theta(\omega)}$ we can write the real and imaginary parts of eq. (1) as

$$\dot{R}(\omega) = -\gamma R(\omega)+\int_{-\infty}^{\infty}\Omega(\omega,\omega')g(\omega')R(\omega')\cos[\theta(\omega')-\theta(\omega)]d\omega' \tag{2}$$

$$\dot{\theta}(\omega) = \omega + \int_{-\infty}^{\infty}\Omega(\omega,\omega')g(\omega')\frac{R(\omega')}{R(\omega)}\sin[\theta(\omega')-\theta(\omega)]d\omega', \tag{3}$$

and, if $R(\omega)$ goes to a steady state in the long-time limit, eq. (3) is simply the Kuramoto model with a generalized coupling

$$K(\omega,\omega') = \Omega(\omega,\omega')\frac{R(\omega')}{R(\omega)}. \tag{4}$$

Note that the variable $R(\omega)$ is introduced simply to carry out the mapping, and has no physical significance. γ is set such that $R(\omega)$ will reach a steady state.

Having reformulated the Kuramoto model in terms of linear dynamics, we can proceed to analyze and solve it using tools from the linear repertoire. Indeed, the synchronization problem can be discussed in terms of the spectrum of the linear operator on the RHS of eq. (1). More precisely, let $\mathcal{K}(\omega,\omega') = \Omega(\omega,\omega')g(\omega') - i\omega\delta(\omega-\omega')$ and assume that the Fredholm integral equation

$$\int_{\Re} d\omega' \mathcal{K}(\omega,\omega')\phi_\sigma(\omega') = \mu_\sigma \phi_\sigma(\omega), \quad \sigma \in \mathbb{Z}, \mathbb{R} \tag{5}$$

has a mixed, discrete-continuum spectrum $\{\mu_n, \mu_\sigma\}$. Then a generic solution of (1) is given by

$$\psi(\omega,t) = \sum_{n\in\mathbb{Z}} a_n \phi_n(\omega) e^{(\mu_n-\gamma)t} + \int_{\sigma\in\mathbb{R}} b(\sigma)\phi_\sigma(\omega) e^{(\mu_\sigma-\gamma)t} d\sigma, \tag{6}$$

with coefficients $\{a_n\}_{n\in\mathbb{Z}}, \{b(\sigma)\}_{\sigma\in\mathbb{R}}$ determined by initial conditions.

The spectrum $\{\mu_n, \mu_\sigma\}$ determines the appropriate value of γ, which we set equal to the real part of the eigenvalue with the largest real part. It becomes evident that this spectrum dictates the synchronization behavior of the system. If there is only one eigenvalue whose real part equals γ then in the long-time limit contributions from all other eigenvalues die away, $R(\omega)$ goes to a steady state, the linear model maps onto the Kuramoto model with time-independent coupling, and there is full phase locking and synchronization (as defined below). Otherwise, more than one eigenvalue remains, $R(\omega)$ does not reach a steady-state value, and the phases of ψ do not converge.

As mentioned earlier, we will restrict our discussion to systems with no partial population of drifting oscillators, i.e. the incoherent-to-partially locked (usually referred to as the synchronization transition) and the partially locked-to-fully locked phase transitions occur at the same point [4]. We say our system is synchronized if the synchronization order parameter given by

$$r = \left| \int_{-\infty}^{\infty} d\omega g(\omega) e^{i\theta(\omega)} \right| = \left| \int_{-\infty}^{\infty} d\omega g(\omega) \frac{\psi(\omega,t)}{|\psi(\omega,t)|} \right| \tag{7}$$

goes to a nonzero steady-state value [2][3]. As mentioned above, this happens at the critical point where the real part of more than one eigenvalue becomes equal to γ. So in the synchronized region where the real part of only one eigenvalue equals γ,

$$r = \left| \int_{-\infty}^{\infty} d\omega g(\omega) \frac{b(\omega)}{|b(\omega)|} \right| \tag{8}$$

where $b(\omega)$ is the eigenfunction corresponding to that differentiated eigenvalue, λ_N.

III. Solution for specific coupling

A. Solution of the synchronization order parameter

Let us consider now one type of global coupling in the linear model, $\Omega(\omega,\omega') = \Omega$. The linear model describes this system as

$$\dot{\psi}(\omega,t) = (i\omega-\gamma)\psi(\omega,t) + \Omega \int_{-\infty}^{\infty} g(\omega')\psi(\omega',t)d\omega', \tag{9}$$

which maps onto the original Kuramoto model with the following coupling constant:

$$K(\omega,\omega') = \Omega \sqrt{\frac{(\omega-\omega_r)^2+\gamma^2}{(\omega'-\omega_r)^2+\gamma^2}}, \tag{10}$$

where ω_r is the collective frequency of the synchronized state and is given by the imaginary part of λ_N, $\Im(\lambda_N)$. With this coupling scheme, we can solve the spectrum of the RHS of eq. (9) exactly.

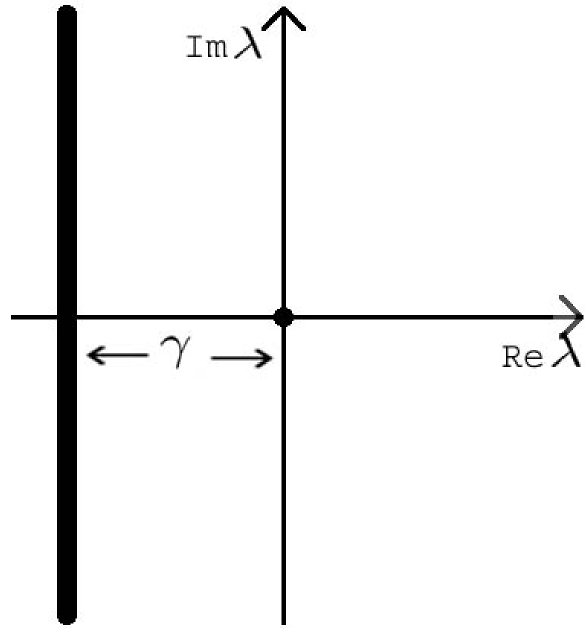

Fig. 1. The spectrum of eigenvalues associated with the RHS of eq. (9) when $\Omega > \Omega_c$. The spectrum comprises a continuum of eigenvalues along $-\gamma$ and a single eigenvalue at the origin. As $\Omega \to \Omega_c^+$, $\gamma \to 0$, and the continuum approaches the imaginary axis and the eigenvalue at the origin. When $\Omega \leq \Omega_c$, $\gamma = 0$, and the continuum of eigenvalues sits on the imaginary axis and the eigenvalue at the origin becomes indistinguishable from the continuum.

With γ set as described above, the spectrum comprises a continuous line of eigenvalues λ in the complex plane along $\Re[\lambda] = -\gamma$ ($\Re[\lambda]$ denoting the real part of λ) for any value of Ω and one eigenvalue λ_N at the origin, which stands apart from the continuum of eigenvalues if $\Omega > \Omega_c$. As $\Omega \to \Omega_c^+$, $\gamma \to 0$, and for $\Omega \leq \Omega_c$ the continuum of eigenvalues lies along the imaginary axis and λ_N becomes indistinguishable from the continuum, as shown in Figure 1; since, in the steady state, the entire spectrum remains, it is clear that $r = 0$ for $\Omega \leq \Omega_c$.

Setting the collective frequency $\Im[\lambda_N]$ to zero, γ to $Re[\lambda_N]$, and assuming $g(\omega)$ is an even function and nowhere increasing for $\omega \geq 0$, we arrive at the following formula that determines

γ:

$$1 = \Omega \int_{-\infty}^{\infty} \frac{g(\omega)\gamma}{\gamma^2 + \omega^2} d\omega \tag{11}$$

By taking $\gamma \to 0^+$ and assuming $g(\omega)$ has a finite width, it becomes clear from eq. (11) that

$$\Omega_c = \frac{1}{\pi g(0)}. \tag{12}$$

From eq. (8), for $\Omega > \Omega_c$, we can determine the following explicit expression for r:

$$r = \int_{-\infty}^{\infty} d\omega g(\omega) \frac{1}{\sqrt{1 + \left(\frac{\omega}{\gamma}\right)^2}}. \tag{13}$$

where γ can be determined from eq. (11). So, for a given distribution $g(\omega)$, eqns (11), (12), and (13) completely specify $r(\Omega)$ and Ω_c for $K(\omega, \omega')$, eq. (10). (For details, see [6].)

It is interesting to note that about $\Omega = \Omega_c$ the behavior of r resembles a second-order phase transition in that r grows continuously from zero as the coupling increases. Where $\Omega \to \Omega_c^+$, $r \to 0$ because, although the oscillators are phase locked, the phases of the oscillators are evenly distributed from zero to 2π, i.e. the system is in a splay state [7]. Below, we investigate the scaling behavior about $\Omega = \Omega_c$, where this second-order "phase transition" occurs.

B. Anomalous scaling

If one assumes $g(\omega)$ to be such that $\int_{-\infty}^{\infty} d\omega g'(\omega)/\omega$ is nonzero and finite (where $g'(\omega) \equiv \partial_\omega g(\omega)$), then perturbatively the behavior of γ as $\Omega \to \Omega_c$ becomes

$$\gamma = -\frac{\pi g(0)}{\int_{-\infty}^{\infty} d\omega g'(\omega)/\omega} \left(\frac{\Omega - \Omega_c}{\Omega_c}\right) + O\left[\left(\frac{\Omega - \Omega_c}{\Omega_c}\right)^2\right]. \tag{14}$$

As $\gamma \to 0$, the behavior of $r(\gamma)$ can be described by

$$r(\gamma) = -2g(0)\gamma \log[g(0)\gamma] + O[g(0)\gamma]. \tag{15}$$

This behavior of r can be seen as anomalous with respect to the usual square-root scaling behavior of the traditional Kuramoto solution. One might venture that the anomalous scaling is due to the bicritical nature of the critical point. For derivation and explanations of this behavior, see [6].

C. Specific examples of characteristic frequency distributions

With these general solutions for the parameters of any system with global coupling $\Omega(\omega, \omega') = \Omega$, we can solve for a specific system given its characteristic frequency distribution. Take for instance the Lorentzian distribution about ω_r, i.e. $g(\omega - \omega_r) = \frac{\Delta}{\pi[\Delta^2 + (\omega - \omega_r)^2]}$. From eq. (12) we find $\Omega_c = \Delta$, and from eq. (11), $\gamma_{lor} = \Omega - \Delta$. Using these in eq. (13), we obtain

$$r_{lor} = \frac{2\cos^{-1}\left(\frac{\Omega_c}{\Omega - \Omega_c}\right)}{\pi\sqrt{1 - \left(\frac{\Omega_c}{\Omega - \Omega_c}\right)^2}} \tag{16}$$

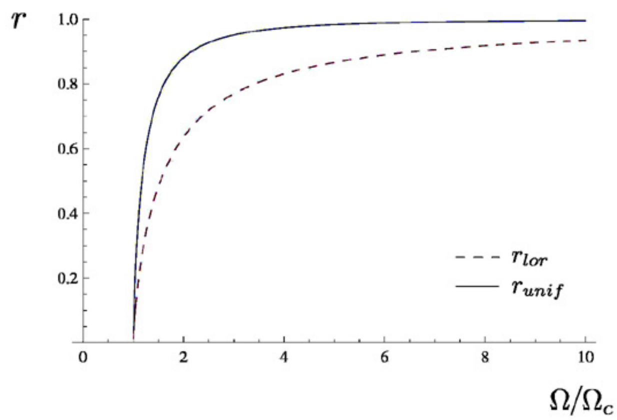

Fig. 2. The synchronization order parameter as a function of the normalized coupling constant for a uniform and Lorentz distribution of characteristic frequencies

for $\Omega > \Omega_c$, as shown in Figure 2. $r = 0$ for $\Omega \leq \Omega_c$ as discussed above. The scaling for this distribution is then

$$r_{lor} \approx \frac{2}{\pi} \frac{\Omega - \Omega_c}{\Omega_c} \log\left(\frac{\Omega_c}{\Omega - \Omega_c}\right). \tag{17}$$

This agrees with eqns (14) and (15) knowing that, for this Lorentzian distribution, $g(0) = \frac{1}{\pi\Delta}$ and $\int_{-\infty}^{\infty} d\omega g'(\omega)/\omega = -\frac{1}{\Delta^2}$.

Similarly, for a uniform distribution about ω_r, i.e. $g(\omega - \omega_r) = \frac{1}{\pi\Delta}$ for $|\omega - \omega_r| < \pi\Delta/2$ and 0 otherwise, the above equations give $\Omega_c = \Delta$, $\gamma_{unif} = \frac{\Delta\pi}{2}\cot\left(\frac{\pi\Delta}{2\Omega}\right)$, and

$$r_{unif} = \cot\left(\frac{\pi}{2}\frac{\Omega_c}{\Omega}\right)\sinh^{-1}\left[\tan\left(\frac{\pi}{2}\frac{\Omega_c}{\Omega}\right)\right] \tag{18}$$

for $\Omega > \Omega_c$ (see Figure 2). As above, $r = 0$ for $\Omega \leq \Omega_c$. The scaling for the uniform distribution is

$$r_{unif} \approx \frac{\pi}{2} \frac{\Omega - \Omega_c}{\Omega_c} \log\left(\frac{\Omega_c}{\Omega - \Omega_c}\right). \tag{19}$$

Again, there is agreement with eqns (14) and (15) as $g(0) = \frac{1}{\pi\Delta}$ and $\int_{-\infty}^{\infty} d\omega g'(\omega)/\omega = -\frac{4}{(\pi\Delta)^2}$.

IV. Conclusion

The linear reformulation presented here of the Kuramoto model constitutes a fresh take on the problem of self-synchronization of a heterogeneous population of coupled phase oscillators, opening it to solution through established linear approaches such as spectral theory. This alternative treatment of spontaneous synchronization can give new insight into the mechanics underlying the phenomenon. In addition, this method allows for analytical solutions of systems with finite oscillator populations. Although we have restricted ourselves to the fully locking transition with global coupling, this method holds great promise for solving partial synchronization states and for synchronization problems in complex topologies.

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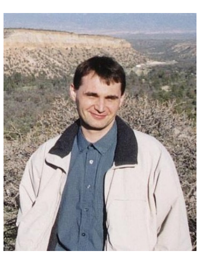

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