Cumulant Analysis of Rössler Attractor

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***Abstract*— This paper is dedicated to the cumulant analysis of the Rössler attractor, based on the so-called "degenerate cumulant equations" method. The approach is ilustrated by the calculus of the first cumulants, which are necessary to create an approximation of the probability density function (PDF), applying the Gramm-Charlier series, the model distribution method, etc. An approximate method for the variance calculation at the output of the Rössler strange attractor is shown. The latter is based on the Kolmogorov-Sinai entropy that is defined by the Lyapunov exponents for a statistically linearized chaotic system and by the differential Kolmogorov entropy.**

***Index Terms*— Cumulants, cumulant brackets, attractor, kurtosis, variance, Kolmogorov-Sinai entropy.**

I. Introduction

Nowadays, the number of chaos applications has grown considerably [1], but there are still a lack of the effective tools for the statistical analysis of the chaotic behavior for strange attractors, particularly in the electrical engineering field.

Recently it was proposed a so-called "degenerate cumulant equations method" [2-4] for applied statistical analysis of the strange attractors, based on the parameters of the corresponding dynamic systems[1]. It was shown [2-4], that by means of the proposed approach not only the expressions for the cumulants can be found, but also the so-called "model distributions" for each component of the attractors under analysis, etc.

The "attractive" features of the cumulants (instead of moments) for engineering purposes was explained in detail at [3], [5] and also a comprehensive and adequate method for tha cumulant calculus was presented: the cumulant brackets.

Note, that the "weight" of cumulants diminish as its order grows [5], so for engineering analysis it is sufficient to consider only the first four cumulants: $\chi_1 - \chi_4$, and corresponding shape coefficients for PDF: γ_3 is the asymmetry coefficient and γ_4 is the kurtosis coefficient [5].

It was also shown recently, that chaotic models are adequate to model several natural phenomena, related to the communication field. For example, output signals of some components of the well known attractors: Lorenz, Chua, etc, can successfully describe the PDF's of the interferences from some digital interconnects, and the cumulant analysis provides a very good coincidence with the measurement data.

The improvement of the characteristics of the modern digital communications from mobile to mobile PC's users, etc is so important, that it is highly encouraging to continue with the statistical research of other attractors, for example, Rössler attractor.

It is worth to notice here, that any cumulant analysis of the dynamic system is not only qualitative, but also quantitative.

As it was stated from the very beginning of its applications, the cumulant method presents a "general view" of the statistical system behavior due to system parameters [6], [7].

As it follows from the title, this paper is dedicated to the cumulant analysis of the Rössler attractor from the engineering point of view. The paper is organized as follows. Section II contains the basics of the degenerate cumulant method and the concept of cumulant brackets. Section III is dedicated to the analytical cumulant analysis of the Rössler attractor. In section IV some of the numerical results for Rössler attractor, their comparison with analytical predictions and the concept of equations for cumulants and cumulants brackets are presented. Section V presents the analytical approximate method for the variance evaluation of the Rössler attractor. Section VI is dedicated to some comments to the section V. Conclusions are presented in section VII.

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[1]Certainly, all statistical properties of the attractor can be obtained through statistical computer simulation, but it is not the case here.

II. Degenerate cumulant equations

It is well known [8], that each dissipative continuous time dynamic system (strange attractor) can be defined with the following equation:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t)), \quad \mathbf{x} \in \mathbf{R}^n, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \quad , \tag{1}$$

where $\mathbf{f}(\cdot) = [f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})]^T$ is a differentiable vector function.

Following the principles for the ergodic theory of the equation (1) [8], one has to annex the external weak noise at (1) (idea of Kolmogorov) in order to obtain the meaningful concept for the physical measure of $\mathbf{x}(t)$ − $W_{st}(\mathbf{x})$. $W_{st}(\mathbf{x})$− is a stationary PDF for $\mathbf{x}(t)$ (see details in [8] and in [3], [4] as well).

Note, that $W_{st}(\mathbf{x})$, as well as its characteristic function, is totally defined by the complete set of cumulants [5].

Let us consider in the following only the one dimensional PDF $W_{st}(\mathbf{x})$ as:

$$\begin{array}{c} W_{st}(x) = F^{-1}\{\theta_m(j\mho)\} \quad , \\ \theta(j\mho) = exp\{\sum_{s=1}^{\infty} \frac{(j\mho)^s}{s!}\chi_s\} \end{array} \tag{2}$$

where χ_s is the cumulant of the $s-th$ order.

If one assumes that cumulants for all $s > m$ are equal to zero, then for the finite set of cumulants $\{\chi_s\}_1^m$, we can introduce the "model distribution" $\hat{W}_{st}(\mathbf{x})$ of the $m-th$ order and its characteristic function is defined by $\theta_m(j\mho)$.

It is clear that $F\{\cdot\}$ and $F^{-1}\{\cdot\}$ are direct and inverse Fourier transforms respectively.

The distribution $\hat{W}_{st}(\mathbf{x})$ is only an approximation of the true PDF and the model distributions ([11], and references therein) provide an accuracy, better than the orthogonal series expansions for the case $\gamma_4 < 0$, and can also be applied for $\gamma_4 > 0$.

Another option for analytical approximation are the orthogonal representations, for example, Gramm-Charlier, Laguerre series, etc. [5]. For example, the Gramm-Charlier series are defined by:

$$W(x) = W_G(x)[1 + \frac{\gamma_3}{3!}H_3(x) + \frac{\gamma_4}{4!}H_4(x)], \tag{3}$$

where

$$H_n(x) = (-1)^n exp(\frac{x^2}{2})\frac{d^n}{dx^n}exp\{-\frac{x^2}{2}\};$$

$H_n(\cdot)$ is a Hermitian polynomial of n-order,

$$W_G(x) = \frac{1}{\sqrt{2\pi}}exp\{\frac{-x^2}{2}\}$$

is a Gaussian distribution,

$$\gamma_3 = \frac{\chi_3}{\sigma^3}$$

is the skewness coefficient and

$$\gamma_4 = \frac{\chi_4}{\sigma^4}$$

is the kurtosis coefficient.

It is important to mention that for a symmetrical PDF the above coefficients satisfy:

$$\gamma_3 = 0, \mu_3 = \chi_3, \mu_4 = \chi_4 + 3(\sigma^2)^2$$

with χ_3 and χ_4 being the third and fourth cumulants; $\gamma_4 \geq -2$.

Now (1) can be rewritten in the form of the stochastic differential equation (SDE):

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t)) + \varepsilon\xi(t) \quad , \tag{4}$$

where $\xi(t)$ is a vector of a weak external white noise with the related positive defined matrix of "intensities" $\varepsilon = [\varepsilon_{ij}]^{nxn}$, [3], [5].

In other words $\mathbf{x}(t)$ is a continuous n-dimensional Markov process with kinetic coefficients, given by $\mathbf{K}_{1_i}(\mathbf{x}) = \mathbf{f}_i(\mathbf{x})$ and $\mathbf{K}_2 = [\epsilon_{ij}]^{nxn}$,[2] [3], [5].

For the SDE representation of the attractor the approach, named as cumulant equations for the SDE with the given $\mathbf{K}_{1_i}(\mathbf{x})$ and $\mathbf{K}_2$ [5], can be successfully applied.

For the moment we assume (as it was proposed above), that $W_{st}(x)$ exists, and that it is a reasonable physical measure for (1) and (4) [13]; although there exists all cumulants that adequately represent $W_{st}(x)$. Those cumulants can be found from the following equations for the stationary cumulants (the interested reader can find all necessary developments in details and with examples at [5, *ch.* 4]):

$$\begin{array}{c} \langle \mathbf{K}_{ij}(\mathbf{x})\rangle = 0, \\ 2\{\langle x_i, \mathbf{K}_{1i}(\mathbf{x})\rangle\}_s + \langle \mathbf{K}_{2ij}\rangle = 0, \\ 3\{\langle x_i, x_j, \mathbf{K}_{1\beta}(\mathbf{x})\rangle\}_s + 3\{\langle x_1, \mathbf{K}_{2j\beta}\rangle\}_s = 0, \\ \vdots \\ \sum_{l=1}^{n} C_n^l[\{\langle x_1, x_2, \ldots, x_{n-l}, \mathbf{K}_{1n-l+1(\mathbf{x})}\rangle\}_s \ldots \\ +\langle x_1, x_2, \ldots, x_{n-l}, \mathbf{K}_{2n-l+1,l}\rangle = 0] \end{array} \tag{5}$$

where $i \neq j, \beta = \overline{1, n}.$

We can evidently see from (5), that if $\forall \epsilon_{ij} \rightarrow 0$, then the second summand in (5) tends to zero and the equations in (5) tend to the so-called "*degenerate cumulant equations*".

Hence, the degenerate cumulant equations have the following form:

[2]Here we apply the definition for kinetic coefficients in Stratonovich form.

$$\begin{aligned}\langle \mathbf{K}_{1_i}(\mathbf{x})\rangle &= 0,\\ 2\{\langle x_i, \mathbf{K}_{1_j}(\mathbf{x})\rangle\}_s &= 0,\\ 3\{\langle x_i, x_j, \mathbf{K}_{1\beta}(\mathbf{x})\rangle\}_s &= 0,\\ &\vdots\end{aligned} \tag{6}$$

where $i, j, \beta = \overline{1, n}$, and $\langle x_i, x_j, \ldots, x_\beta\rangle$ are the so-called "*cumulant brackets*" - abbreviated representation for any cumulant [5].

$A\{x, y, \ldots, z\}_s$ is an abbreviation of the Stratonovich symmetrization brackets (A is an integer) and represents the sum of all possible permutations in times of the arguments inside the brackets [5] (see Appendix).

Essentially equations (5) and (6) represent a set of non-linear algebraic equations and this set, in general, is not closed, but it is always possible to cut the set of cumulants by neglecting all cumulants with order $s > m$.

The equations (6) have to be sequentially solved first for each component of $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ (first line); next for couples of components $\{x_i, x_j\}_{i,j=1}^{n}$ (second line), and then for triplets $\{x_i, x_j, x_\beta\}_{i,j,\beta=1}^{n}$ (third line), etc.

The way to do it is to "open" the cumulant brackets as shown in Appendix A2 at chapter 4 at [5].

To illustrate the procedure described above, we apply this material to the analysis of the Rössler attractor.

III. Rössler attractor

Equation (1) for Rössler attractor has the following form:

$$\begin{aligned}\dot{x} &= -y - z\\ \dot{y} &= x + ay\\ \dot{z} &= b + zx - zc\end{aligned} \tag{7}$$

where a, b, c are the parameters of the attractor, $\mathbf{x} = [x, y, z]^T$.
Thus, the first kinetic coefficients for (6) are:

$$\begin{aligned}\mathbf{K}_{1,1}(x) &= -y - z\\ \mathbf{K}_{1,2}(x) &= x + ay\\ \mathbf{K}_{1,3}(x) &= b + zx - zc.\end{aligned} \tag{8}$$

Introducing (7) and (8) into (6) for the first component "x", one gets:

$$\left.\begin{aligned}\chi_1^y = \langle y\rangle = -\langle z\rangle &= -\chi_1^z\\ \chi_{1,1}^{x,z} &= -\chi_{1,1}^{x,y}\\ \chi_{2,y}^{x,y} &= -\chi_{2,1}^{x,z}\\ \chi_{3,1}^{x,y} &= -\chi_{3,1}^{x,z}\\ \chi_{4,1}^{x,y} &= -\chi_{4,1}^{x,z}\end{aligned}\right\} \tag{9}$$

For the second component "y":

$$\left.\begin{aligned}\chi_1^x = \langle x\rangle = -a\langle y\rangle &= -a\chi_1^y\\ \chi_2^y &= -\frac{1}{a}\chi_{1,1}^{x,y}\\ \chi_3^y &= -\frac{1}{a}\chi_{2,1}^{y,x}\\ \chi_4^y &= -\frac{1}{a}\chi_{3,1}^{y,x}\end{aligned}\right\} \tag{10}$$

For the third component "z":

$$\left.\begin{aligned}\chi_{1,1}^{x,z} = \chi_{1,1}^{z,x} = c\langle z\rangle &= -\langle z\rangle\langle x\rangle\\ \chi_2^y &= \frac{c\langle z\rangle - a\langle z\rangle^2}{a}\end{aligned}\right\} \tag{11}$$

Then, going on with two components $\{x, y\}$, one can get: $\chi_3^y = \frac{2\chi_{1,2}^{y,x}}{1+a} \neq 0$, i.e PDF of $y(t)$ is asymmetrical.

With two components $\{x, z\}$ it follows: $\chi_3^z \neq 0$, i.e PDF of $z(t)$ is asymmetrical as well, and

$$\chi_2^y = \frac{\chi_2^x}{\langle z\rangle} \tag{12}$$

Please note, that notations for cumulants described in this section are considered in the Appendix.

In the same way all necessary cumulants of the higher order can be found.

IV. Numerical Results and discussions

The results were obtained using Matlab. It is pretty simple to carry out numerical simulations of any strange attractor, especially Rössler attractor, and find all the cumulants (moments) necessary to make analytical approximation of the PDF for each component, etc.

Let us consider the results of simulations of the Rössler attractor with the following parameters: $a = 0.2$, $b = 0.2$, $c = 5.7$ [6], [7].

Note, that obviously each component of the attractor achieves its stationary conditions with different time constant. At this step of the research it was empirically found (from simulations), that the "x" components reach their stationary state faster than "y and z" components. Therefore for comparison with the analytical results it is reasonable to apply data from the simulations, when the PDF's clearly achieve their stable shapes.

Keeping this in mind, let us make brief comparisons between analytical and simulation results. Based on the analytical results, which were presented in section II, we'll consider mainly the comparisons of the first two cumulants, as they are very important for practice.

From simulations it was found:

$$\langle x\rangle = 0.17, \quad \langle y\rangle = -0.76, \quad \langle z\rangle = 0.75.$$

One can see from (9) and (10) an almost exact coincidence with theory.

It is important to mention that all the components of the Rössler attractor are normalized before realizing the calculations.

Then, from (11) it follows, $\chi_2^y = 17.8$ and simulation gives $\chi_2^y = 18.9$ (error is about 5.8%).

From (10) $\langle x\rangle = 0.13$ and $|\langle x\rangle| < \langle y\rangle|$: and $\langle x\rangle = 0.17$, from (12) it comes: when $\chi_2^x = 13.3$, $\chi_2^y = 17.8$ (simulation gives almost the same.)

Both PDF's for components "y" and "z" are not symmetrical. From (10) it follows, that PDF's for "x" and "y" are oppositely asymmetric, in contrary to Lorenz attractor (see [3]).

It can be seen, that besides of a good accuracy of the analytical prediction for cumulants, for complete calculus of the cumulants it is mandatory to provide the calculus of the variances: $\chi_2^x, \chi_2^y, \chi_2^z$, as high order cumulants depend to them.

In Figure (1-3) are represented the histograms for PDF's of the "x", "y" and "z" components of the Rössler attractor.

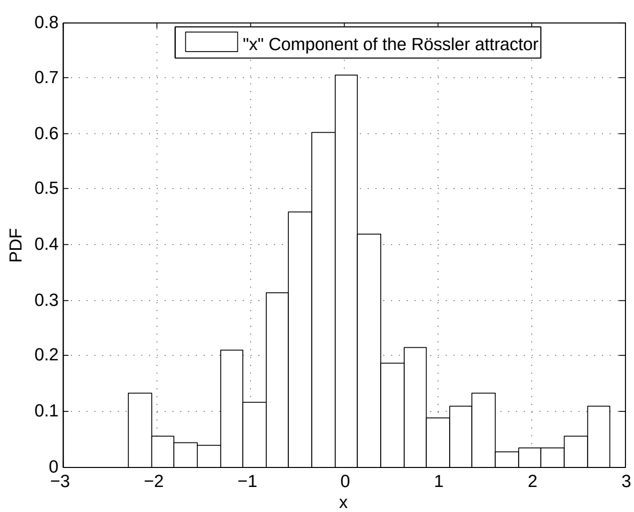

Fig. 1. "x" component of the Rössler attractor

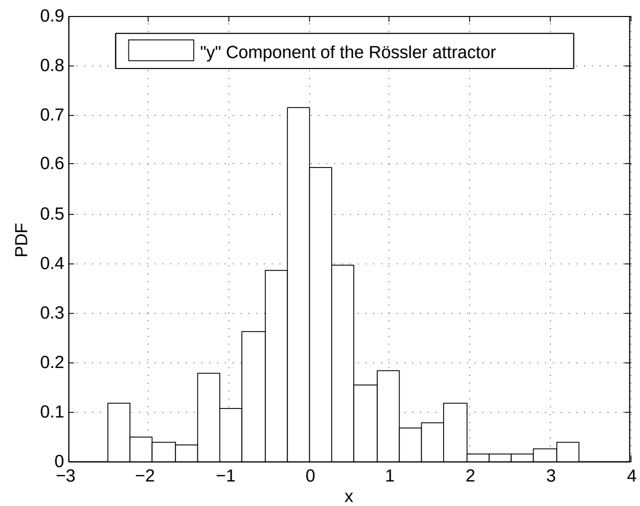

Fig. 2. "y" component of the Rössler attractor

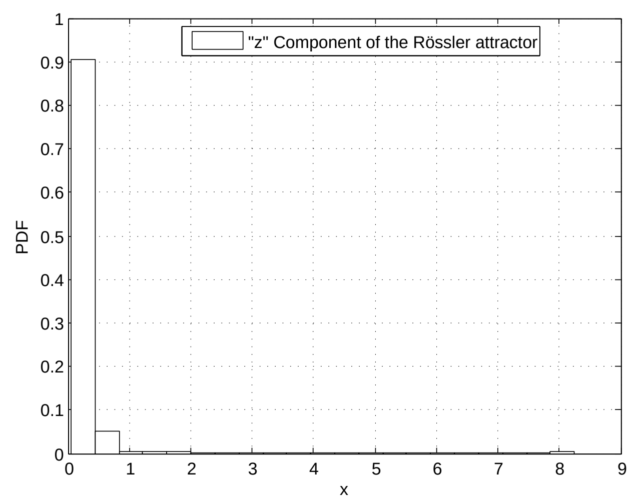

Fig. 3. "z" component of the Rössler attractor

One can see, that the components "x" and "y" are "oppositely" symmetric (see also (8)), and have unimodal PDF'S with $\gamma_4 > 0$, i.e. the vertices of the distributions are "sharper", than the Gaussian ones.

We can see that the component z can be approximated by means of a delta- function.

For "x" and "y" components we describe the PDF histograms by means of the Laplace distribution defined by:

$$f(x) = \frac{1}{2\lambda} exp\{-\frac{|x-\mu|}{\lambda}\}.$$

This distribution is characterized by location μ (any real number) and scale λ (has to be greater than a 0) parameters. The use of Laplace distribution allows to make a right description for the "x" component and "y" components of the Rössler attractor as it is observed in Figures 4 and 5.

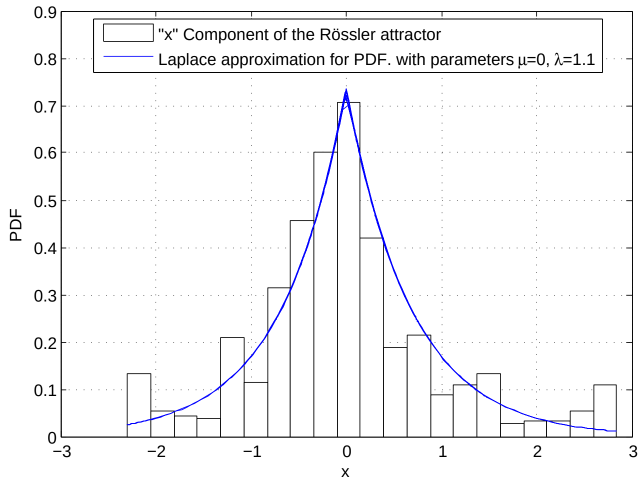

Fig. 4. "x" component of the Rössler attractor

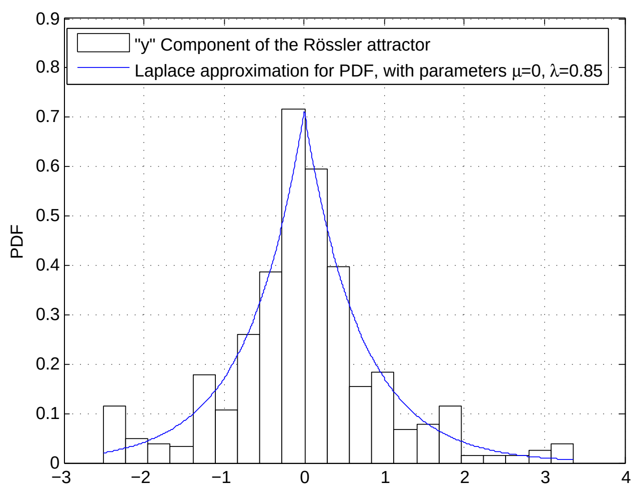

Fig. 5. "y" component of the Rössler attractor

Then, as it follows from Figures 4 and 5, the (PDF), for the "x" component of the Rössler attractor is approximated by a Laplace distribution with the local parameter $\mu = 0$ and scale parameter $\lambda = 1.1$ and for the "y" component with local parameter $\mu = 0$ and scale parameter $\lambda = 0.85$.

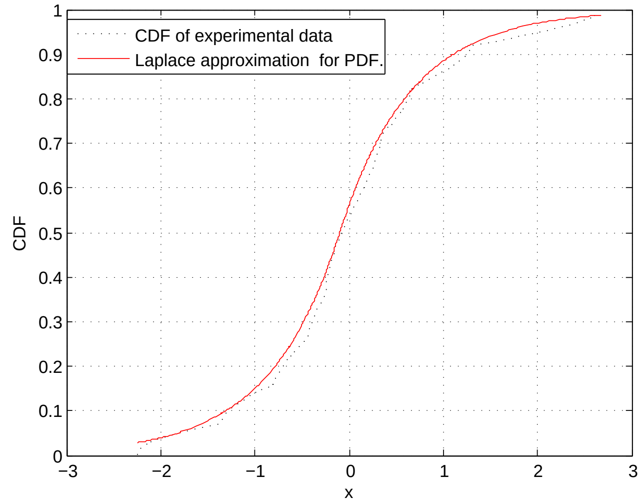

Fig. 6. CDF of the "x" component using the Laplace approximation

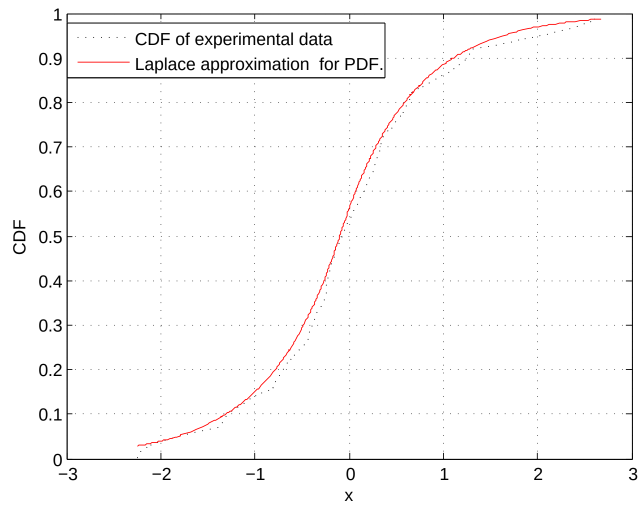

Fig. 7. CDF of the "y" component using the Laplace approximation

In Figures 6 and 7, we apply the Kolmogorov-Smirnoff goodness of fit test with a significance level $\alpha = 0.05$, in order to examine whether the accuracy of the PDF of the "x" and "y" components of the Rössler attractor and its approximation using the Laplace distribution are adequate or not. As it can be seen from the figures 6 and 7, the approximation can be considered as aceptable.

In Figures 8 and 9, the results of the approximation of the PDF for the "x" and "y" components of the Rössler attractor by means of the Gramm-Charlier series and the model distribution (2) (both with$\gamma_2 > 0$,) are represented.

One can see, that, as it was commented before, the Gramm-Charlier approximation is more precise, than the expression (2), because of $\gamma_2 > 0$.

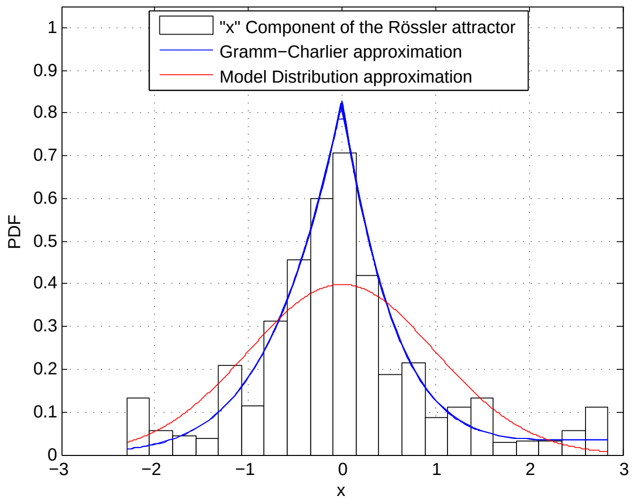

Fig. 8. The "x" component of the Rössler attractor with the Laplace and model distribution approximations.

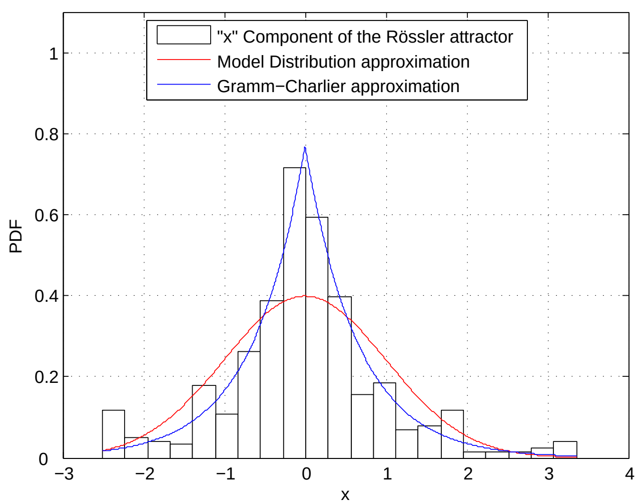

Fig. 9. The "y" component of the Rössler attractor with the Laplace and model distribution approximations.

V. Approach for Variance Calculation

By considering the results presented in previous sections we realize that the approximations obtained by applying the cumulant method are really good.

As it is emphasized above, the first four cumulants for the Rössler attractor depend on the evaluation of the variance χ_2 that is based on the Kolmogorov-Sinai H_{k-s} entropy.[3] The H_{k-s} entropy is defined by the sum of the Lyapunov exponents for a non-linear system [9, page 122]. For a linear matrix the Lyapunov exponents are defined by its eigen-values [9, pages 542-543].

In the proposed methodology we compare the differential entropy obtained from the PDF approximation for a component of the Rössler attractor, with the H_{k-s} entropy computed through the parameters of the attractor (Lyapunov exponents).

Note, (see in the following) that in the framework of this analysis H_{k-s} coincides with the Kolmogorov differential entropy of the PDF [9, pages 542-543, 839].

The proposed methodology can be summarized as follows.

1) The non-linear system describing the chaotic behavior of the Rössler attractor has to be statistically linearized [9, p 760].
2) The eigen-values must be found from the coefficients matrix formed by the linearized system.
3) Once the eigen-values have been obtained, the H_{k-s} for the dynamic system can be estimated as follows:

$$log|\lambda_{max}| < H_{k-s} \le \sum_{j=1}^{m} log|\lambda_j|, \tag{13}$$

   where, $j = \overline{1, m}$, λ_j - is the $j-th$ eingen-value of the linear matrix and $log|\lambda_j|$ - is the $j-th$ Lyapunov exponent for the linear matrix [9, p 542].
4) On the other hand the Kolmogorov differential entropy is:

$$h_{dif} = -\int_{\infty}^{-\infty} W(x)log[W(x)]dx, \tag{14}$$

   where $W(x)$ is the PDF of the ouput signals whose parameters have to be represented through χ_2.
5) Then, we create an algebraic equation depending on the variance according to steps 3 and 4.
6) Solving equation from steps 3 and 4, we can have now a solution for variance.

It is worth mentioning, that as the whole Kolmogorov-Sinai entropy is addressed to the given component of the Rössler attractor the value of the variance obtained from the approach is actually its upper bound. The lower bound can be found applying the left hand side of the inequality (12).

Here we present the results of the application of this method for the Rössler attractor considering only the x component. From equation (5) we can obtain a linearized system as it was mentioned in step 1. From step 2, we obtain the coefficient matrix **A** of this linearized system as:

[3] Considering that K-S entropy can't be exactly calculated, it is impossible by using this method to obtain an exact result for variance. Nevertheless, assuming that for practical purposes an error about 10-20% is acceptable, we are able to apply this method.

$$\mathbf{A} = \begin{vmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{vmatrix}$$

Matrix **A** has a determinant different from zero, for this reason we can obtain eingen-values, considering $a = 0.2$ and $c = 5.7$. The characteristic equation becomes:

$$\lambda^3 - (a - c)\lambda^2 - (ac - 1)\lambda + c = 0,$$

where $\lambda_1 = -5.7$, $\lambda_2 = 0.1 + j$ and $\lambda_3 = 0.1 - j$.

Note, that the main eigen-value λ_1 is equal to the parameter c.

Having the results for the eigen-values, and substituting λ_1 into (12), we can obtain a numeric value for the entropy as $H_{k-s} \approx 1.74$.
And for steps 3 and 6 it follows:

$$\chi_2^x = \frac{exp(3.48)}{2e^2} = 2.1.$$

From the Fig. 4 it follows, that $\chi_2^x \approx 2$ and the error is 5%.

A. Some Modifications

In the previous material it was assumed that the parameters of the Rössler strange attractor are predefined as well as the PDF's for the output signals.

Here we'll consider a more general case. Let us suppose, that the set of parameters of the strange attractor and the PDF's are not predefined, but the chaotic regime of the attractors is established.

Then one can apply for the PDF choice the so-called "Maximun Entropy Method (MEM)" [10]. If the attractor of interest can be characterized by the symmetric PDF's, then it can be applied as a MEM distribution Gaussian PDF.

From the MEM principle it follows, that for the given H_{k-s} for Gaussian PDF, the evaluated χ_2 has to be a lower boundary for its true value.

As it was mentioned before if one of the positive Lyapunov exponents predominates it is possible to apply for the lower boundary for χ_2 the inequality $H_{k-s} > log|\lambda max|$. If all values for positive Lyapunov exponents are comparable, then for the lower boundary evaluation it is possible to assume $\frac{1}{3}H_{k-s} \leq \sum_i log|\lambda_i|$, if the strange attractor consists of three equations and we assume that all components are statistically independient.

It follows from (12), that the variance χ_2 obeys the following inequality:

$$\chi_2 \geq \frac{10^{2H_{k-s}}}{2\pi e} \tag{15}$$

For the Rössler attractor $\chi_2^x \geq 1.8$ (true value is 2)

VI. Some comments of the section V

The material of the section V shows that through the ordinary equations of strange attractors and their statistical linearization it is possible to obtain analytical estimations of the Lyapunov exponents of the non-linear dissipative system describing the attractor. After the Lyapunov exponents are found it is possible to get a well known approximation for the Kolmogorov-Sinai entropy H_{k-s}.

On the other side, applying the attractors PDF's for the components of interest (from computer simulations) together with its analytical approximation, it is possible to find the Kolmogorov differential entropy (h_{dif}) wich practically coincides with H_{k-s} for the strange attractors under consideration. Finally, by solving a simple quadratic algebraic equation one can find $\chi_2 \geq 0$.

The approach shows acceptable accuracy for engineering evaluation of χ_2 (less than 20%) and it is completely analytical. Why it happened?

This means that it has to depend on the details of the stochastic set, dimension of the phase space, etc. (see [9,12], etc. for details) of the attractor.

For example if the dimension of the stochastic set D for the strange attractor is more that two, almost all phase trajectories, that constitute the strange attractor are localized in a very thin layer, i.e. it can be approximately represented by a one dimensional Poincare mapping (details can be found in [12]). It is actually true for Chua, Lorenz, Rössler etc. strange attractors as well (see [12], [14], etc).

So, any non-linear dissipative system with a dimension D equal or more than two represents a one-dimensional Poincare mapping (one dimensional dynamics) and one of the consequences of this is the similarity between H_{k-s} and the Kolmogorov differential entropy. Therefore, the acceptable accuracy of the proposed approach might be, in some sense, predicted.

Next, please, note that the dependence of χ_2^x on the parameters of the strange attractors (even in the framework of the predefined chaotic regime) is not trivial and it depends on the attractor's type. As it was shown above, the χ_2^x for the Rossler attractor clearly shows the dependence on the unique parameter c (see also [10]).

VII. Conclusions

1) A rather simple method for statistical analysis of the Rössler attractor, based on the "degenerate cumulant equations" was presented.
2) It was shown, that application of the cumulant analysis gives not only a qualitative picture of

the stochastic behavior of the attractor, but also a quantative evaluation for cumulants.

3) Cumulants can be useful to obtain an approximation of the plot of PDF's of output signals for Rössler attractor.
4) The solution for variance for Rössler strange attractors presented in Section V results to be appropriate as a prediction for this parameter. The error in the calculation is considered permissible in the estimation approach as well.

Acknowledgment

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Appendix

The concept of cumulant brackets was introduced as an abbreviated representation for any cumulant, i.e.

$$\chi^{\xi_1,\xi_2,\ldots,\xi_n}_{m_1,m_2,\ldots,m_n} = \langle \underbrace{\xi_1,\ldots,\xi_2}, \underbrace{\xi_2,\ldots,\xi_2},\ldots, \underbrace{\xi_n,\ldots,\xi_n} \rangle \equiv \langle \xi_1^{[m_1]}, \xi_2^{[m_2]},\ldots,\xi_n^{[m_n]} \rangle \quad ; \tag{16}$$

where ξ_1 appears inside the brackets m_1 times, ξ_2 appears m_2 times, and so on; for example, the third cumulant is $K^{\xi_1,\xi_2}_{1,2} = \langle \xi_1, \xi_2, \xi_2 \rangle$. Some useful features for cumulant brackets can be found in [5,11].

The relations between moments and cumulants as well as the relations between moment and cumulant brackets were discussed in a exhaustive way in [5,11]; the fact that by means of cumulants brackets it is possible to formalize the operations between random variables and their transformations (linear and nonlinear, inertial and non-inertial), in a easy way, was presented in [11] (see also [5] for some generalizations).

For the above presented material some features for cumulant brackets need to be taken into account [5,11]:

1) $\langle \xi, \eta, \ldots, \omega \rangle$ is a symmetric function of its arguments.
2) $\langle a\xi, b\eta, \ldots, g\omega \rangle = a \cdot b \cdot \cdots \cdot g \langle \xi, \eta, \ldots, \omega \rangle$, where $a, b, \ldots, g$ are constants.
3) $\langle \xi, \eta, \ldots, \theta_1 + \theta_2, \ldots, \omega \rangle = \langle \xi, \eta, \ldots, \theta_1, \ldots, \omega \rangle + \langle \xi, \eta, \ldots, \theta_2, \ldots, \omega \rangle$.
4) $\langle \xi, \eta, \ldots, \theta, \ldots, \omega \rangle = 0$, if θ is independent of $\{\xi, \eta, \ldots, \omega\}$.
5) $\langle \xi, \eta, \ldots, a, \ldots, \omega \rangle = 0$.
6) $\langle \xi + a, \eta + b, \ldots, \omega + g \rangle = \langle \xi, \eta, \ldots, \omega \rangle$.

In addition to cumulant brackets it is necessary to introduce here the concept of Stratonovich symmetrization brackets: symmetrization brackets together with the integer number in front of the brackets represent the sum of all possible permutations of the arguments inside the brackets. For example, the operator $3\{\langle \xi_1 \rangle \cdot \langle \xi_2, \xi_3 \rangle\}_s$ means that

$$\begin{aligned} 3\{\langle \xi_1 \rangle \cdot \langle \xi_2, \xi_3 \rangle\}_s &= \langle \xi_1 \rangle \cdot \langle \xi_2, \xi_3 \rangle \ldots \\ &+ \langle \xi_2 \rangle \cdot \langle \xi_1, \xi_3 \rangle + \langle \xi_3 \rangle \cdot \langle \xi_1, \xi_2 \rangle \quad , \end{aligned} \tag{17}$$

where $\{\}_s$ is the notation for the Stratonovich symmetrization brackets. Rules for manipulations with cumulant brackets can be found in A2[5].

The relations between moments and cumulants for the same distribution $W(x)$ are well known (see, for instance, references [5,11]).

The following notation is also quite known: $\langle g(x) \rangle$, $\langle \xi_1 \cdot \xi_2 \rangle$, which denotes the operator of statistical average for $g(x)$ and for the product of two random variables ξ_1 and ξ_2, respectively.

Following [5] we call this operator $\langle \cdot \rangle$ as moment brackets. The formal difference between moment and cumulant brackets is, that the first one contains a 'dot' between random variables (usually it is skipped), and the second one contains a 'comma' between variables.

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