Instabilities driven by dipole resonances in cold atoms

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***Abstract*— We present a model for the coupling between the recently predicted hybrid wave modes and the well-established dipole resonances observed in a cloud of cold atoms. We show that such oscillations are described by a forced oscillator equation in the form of a generalized Mathieu equation. We report on the stability conditions. A feedback control scheme is discussed in order to control the unstable solutions of the homogenous system.**

I. INTRODUCTION

In the last years, special attention has been given to the low temperature physics [3]. The advent of laser cooling techniques, essentially because of the growing interest in the Bose-Einstein condensation, has envisage the possibility to explore, both theoretic and experimentally, new exciting features in the field. Many fundamental aspects of low temperature physics, which have recently been reported, arise from several disciplines and may compromise the actual boundaries between them. As example, one should emphasize the studies developed in cold atoms and Bose-Einstein condensation, and very exciting theoretical works in quantum plasmas.

In this work, we explore the nonlinear coupling between the recently predicted hybrid waves and the dipole resonance due to the trapping potential, in a system of cold atoms [6]. The hybrid modes in cold atoms are formally similar to the Langmuir plasma waves [1], [2], but they present an acoustic nature. Therefore, such modes consist on sound oscillations that exhibit a lower cut-off when $k \to 0$.

II. BASIC EQUATIONS AND DERIVATION OF THE MODEL

A system of cold atoms is achieved by mean of a magneto-optical trapping (MOT), which results of the combination of Doppler cooling techniques with the spatial confinement potential due to the magnetic field. In the low density Doppler model, the effective external force $\vec{F}_{MOT} = -\alpha\vec{v} - \kappa\vec{r}$ depends on the experimental parameters $\kappa = \alpha\mu/k_L$, which represents the spring constant of the trap, and α, the friction coefficient, given by $\alpha = -8\hbar^2 s\Delta/\Gamma/(1 + 4\Delta^2/\Gamma^2)$. Here, $s = I_0/I_{sat}$ is the incident on-resonance saturation parameter per beam, I_0 is the incident laser intensity, Γ the natural line width of the transition used in the cooling process and $\Delta = \omega_L - \omega_a$ the frequency detuning between the laser frequency $\omega_L = k_L c$ and the atomic transition frequency ω_a. The spring constant κ defines the natural time scale, the dipole frequency $\omega_D = \sqrt{\kappa/M}$, where M represents the mass of a single atom. The validity of this model is known to be limited to only a moderate number of atoms (typically $10^5 - 10^6$). For larger number of atoms, additional forces need to be taken into consideration. Therefore, the second force to be considered is the shadow force, or absorption force, $\vec{F}_A$, and was first discussed by Dalibard [4]. This is associated with the gradient of the incident laser intensity due to laser absorption by the atomic cloud. It is an attractive force which can be determined by $\nabla \cdot \vec{F}_A = -\sigma_L^2 I_0 c^{-1} n(\vec{r})$, where σ_L represents the laser absorption cross section and $n(\vec{r})$ is the atom density profile. The third force to be included in this model is the repulsive force between the atoms due to the radiation pressure, $\vec{F}_R$, and can be given by $\nabla\cdot\vec{F}_R = \sigma_L\sigma_R I_0 c^{-1} n\vec{r}$, where σ_R represents the scattering cross section. This allows us to describe the dynamics with an effective force $\vec{F}_T = \vec{F}_{MOT} + \vec{F}_c$, where $\vec{F}_c = \vec{F}_A + \vec{F}_R$ may be regarded as a collective self-consistent force, which is given by $\nabla\cdot\vec{F}_c = Qn, \quad Q = \sigma_R(\sigma_L - \sigma_R)I_0 c$. Here Q stands for the effective charge and allows us to regard the system of cold atoms as a one-component plasma, confined in an external potential. The natural time scale associated with this effective fluid description is given by the generalized plasma frequency $\omega_P = \sqrt{Qn_0/M}$, where n_0 is the unperturbed density profile, such that $\nabla\cdot\vec{F}_T(n = n_0) = 0$.

By taking the zeroth and first order momenta in the Fokker-Planck equation, we can write down the system of equations that effectively describe an amount of cold atoms as a fluid,

$$\begin{aligned} \frac{\partial n}{\partial t} + \nabla\cdot(n\vec{v}) &= 0, && (1)\\ \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} &= \frac{\vec{F}_T}{M} - \frac{\vec{\nabla}P}{Mn}, && (2)\\ \nabla\cdot\vec{F}_T &= Q(n - n_0). && (3)\end{aligned}$$

This system has been also used by other authors to describe Coulomb explosions in optical molasses [5]. The principal aim of this work is to describe the coupling between the two time scales given by ω_D and ω_P, which respectively represent the typical time scales of the oscillations of the center-of-mass and the acoustic waves that may be excited in the system. To proceed with this task, we will use the linear response theory techniques by separating each relevant physical quantity into its equilibrium and perturbation components, such as

$n = n_0 + \tilde{n}$, $\vec{v} = \vec{v}_0 + \delta\vec{v}$, $\vec{F}_T = \vec{F}_{MOT} + \delta\vec{F}_c$. Setting the equilibrium velocity field $\vec{v}_0(t) = \vec{u}_0 \sin\omega_D t + \phi$, basic algebra calculations yield

$$\frac{\partial^2 \tilde{n}}{\partial t^2} + n_0 \frac{\partial}{\partial t}\nabla\cdot\delta\vec{v} + \\ +\vec{u}_0\cdot\nabla\left[\frac{\partial\tilde{n}}{\partial t}\sin(\omega_D t+\phi) + \tilde{n}\omega_D\cos(\omega_D t+\phi)\right] = 0, \quad (4)$$

where he have retained only the terms in first order of perturbation. We will later devote some attention to the question of how higher orders in the perturbative analysis could originate qualitatively different results. In particular, we will explain the origin of the saturation in the stability observed by di Stefano *et al.* in Ref. [7]. One of the problems that may arise in this effective model is concerned with the equation of state for the hydrodynamical pressure $P = P_0 + \tilde{P}$. In the present derivation, we will assume that the cloud of cold atoms is described by adiabatic law $P \propto n^\gamma$, where γ represents the adiabatic constant. Therefore, we can define the sound velocity $u_S = \gamma\frac{P_0}{M}$ for the equilibrium system. Regarding that the wave number k of the sound waves is small enough compared to the size of the system, such that it can be regarded as an infinite medium, the following relation holds

$$\frac{\nabla^2\tilde{P}}{P_0} \approx -\gamma\frac{k^2\tilde{n}}{n_0}. \quad (5)$$

By putting eqs. (4)-(5) together, we should obtain

$$\frac{\partial^2 \tilde{n}}{\partial t^2} + (\omega_P^2 + u_S^2 k^2)\tilde{n} + \\ +\vec{u}_0\cdot\nabla\left[\frac{\partial\tilde{n}}{\partial t}\sin(\omega_D t+\phi) + \omega_D t\cos(\omega_D t+\phi)\right] = 0, \quad (6)$$

and assuming that $\tilde{n}(\vec{r},t) = \tilde{B}(\vec{r})\tilde{A}(t)$, we may finally write

$$\frac{\partial^2\tilde{A}}{\partial\tau^2} + \left[\Delta + 2\epsilon\cos(2\tau)\right]\tilde{A} + 2\epsilon\sin(2\tau)\frac{\partial\tilde{A}}{\partial\tau} = 0, \quad (7)$$

where we defined $2\tau = \omega_D t + \phi$, $\Delta = 4(\omega_P^2 + u_s^2 k^2)/\omega_D^2$ and $\epsilon = 2\vec{u}_0\cdot\nabla\ln\tilde{B}/\omega_D$. Equation (7) belongs to the family of Hill equations and is formally similar to the Mathieu equation, whose properties are already well-known. The interest of this work thus remain in the numerical exploration of the homogeneous system in order to understand how to design a feedback system to control the instabilities.

In order to conclude the discussion of the main features of this model, we point out some remarks concerning the first order perturbation used in its derivation. One of the reasons for such an assumption has to deal with the approximation of parabolic potential, related to the confinement force $-k\vec{r}$ discussed in the introduction. In that case, the linear response theory holds. This obviously does not correspond to the experimental reality, since the potential is not perfectly parabolic, leading the instabilities to saturate, as observed in Ref. [7]. Such saturation would be explained by taking higher order of

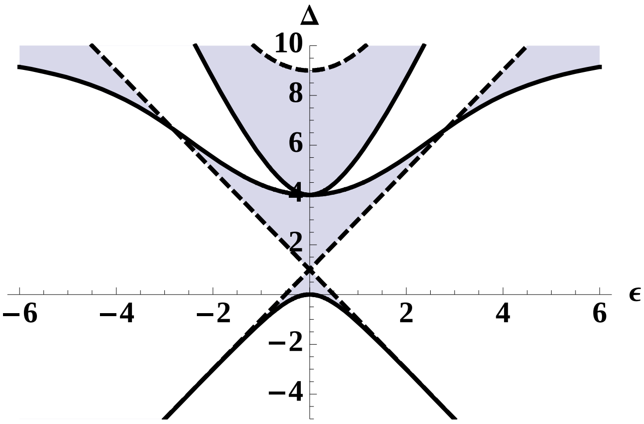

Fig. 1. Characteristic curves separating the stable and the unstable regimes computed numerically by using the Floquet's theory. The shadowed zones correspond to the first three stable zones. The full lines repreent the π-periodic solutions and the dashed lines represent the 2π-periodic ones.

perturbation in Eq. (7), which would introduce anharmonicity to the system and thus an amplitude dependent frequency, leading to the detuning between the natural and the forcing frequencies, saturating the instability. For that reason, this model illustrates the "worse case scenario" of the instabilities that may occur in a magneto-optical trap and so the design of a robust control system may be required. In the future, we may think of extending this techniques to saturated instabilities which, under the experimental point of view, would represent the more interesting problem.

III. Stability and free response of the generalized Mathieu equation

By using the Floquet theory, it is possible to verify that the general solution to eq. (7) is a linear combination of two periodic functions, whose frequencies are given by ω_D and the imaginary part of the characteristic exponent, say γ [8]. Depending on the sign of the $\Re(\gamma)$, one of the solutions is either bounded or unbounded, and so defining the the stability of the general solution. Therefore, if $\Re[\gamma(\epsilon,\Delta)] > 0 \quad (< 0)$, the system is said to be unstable (stable). In fig. (1) we plot the first stability regions in the Strutt chart of (7), for the case of π and 2π-periodic solutions.

Under the physical and experimental point of view, there are three cases of major interest: (a) the wave frequency is much lower than the dipole frequency, $\Delta \lll \omega_D$; (b) the wave and dipole frequencies have the same order of magnitude, $\Delta \approx \omega_D$; and (c) the wave frequency is much higher than the dipole frequency, $\Delta \ggg \omega_D$. However, given the nature of this work, only one case may be enough to motivate a feedback control problem, so we decide to explore the resonant case $\Delta = 2$.

IV. Design of a feedback control system

Equation (7) does not envisage the usual PID methods [9] used to control linear systems. Therefore, we should adopt solutions which are typical of time-dependend or even nonlinear analysis. Depending on the design goals, there are several

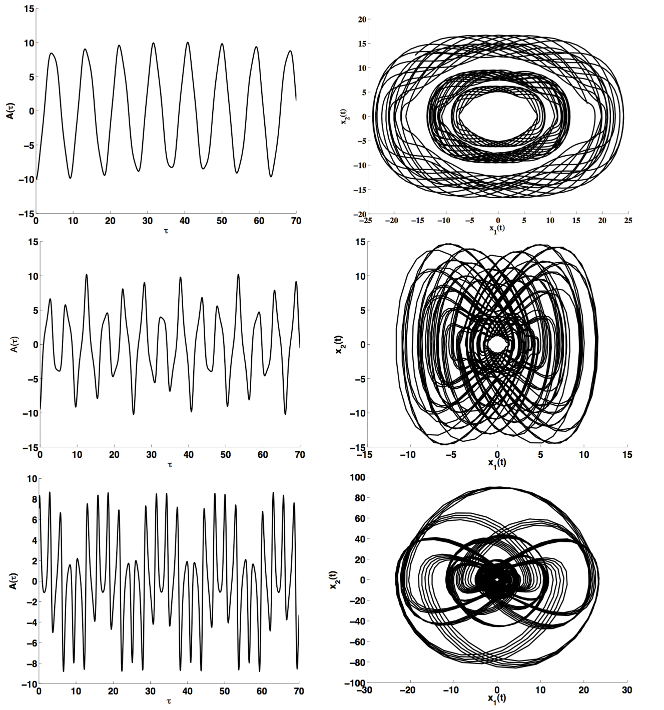

Fig. 2. Time evolution (left) and phase-space portraits (right) of the stable (bounded) solutions of Eq. (7) plotted for different stability zones. From top to bottom: $\Delta = 0.5$, $\epsilon = 0.2$; $\Delta = 2.0$, $\epsilon = 1.0$; $\Delta = 8.0$, $\epsilon = 2.0$.

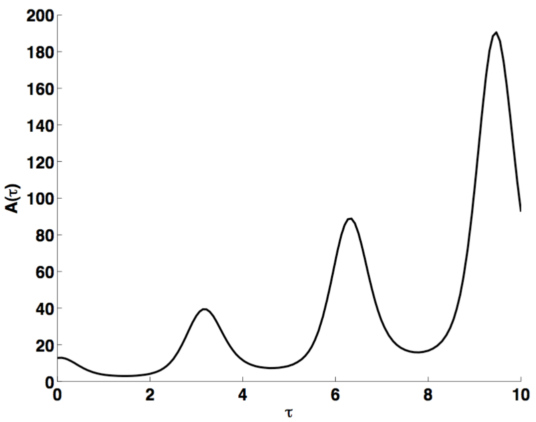

Fig. 3. Unstable solution of eq. (7) ploted for $\Delta = 2$ and $\epsilon = 2.1$.

formulations of the control problem. The tasks of stabilization, tracking and disturbance rejection or attenuation (or even combinations of them) lead to a large number of choices [12]. In each one, one may have a state feedback version where all state variables can be measured or an output feedback version where only few variables can be measured. Other solutions are related rather with stochastic than deterministic methods, like fuzzy-logic control [10]. In a typical control problem, there are additional tasks for the design, like meeting certain requirements on the transient response or certain constraints on the control input, recurrently related to hardware or software limitations. When model uncertainty is taken into account, issues of sensitivity and robustness play an important role. Therefore, the attempt to design a feedback control system to cope with a wide range of uncertainty models leads to either robust or adaptive control systems.

In this work, we will limit our discussion to the case of a disturbance rejection problem. At the end of this section, we devote some attention to the main reasons for that choice.

If we define the output vector $\mathbf{x}(\tau) \equiv (x_1(\tau), x_2(\tau)) = (\tilde{A}(\tau), \dot{\tilde{A}}(\tau))$, we can rewrite (7) in the form

$$\dot{\mathbf{x}}(\tau) = \mathbf{A}(\tau) \cdot \mathbf{x}(\tau), \tag{8}$$

where $\mathbf{A}(\tau)$ represents the matrix of the time-dependent dynamical system

$$\mathbf{A}(\tau) = \begin{bmatrix} 0 & 1 \\ -\Delta - 2\epsilon\cos(2\tau) & -2\epsilon\sin(2\tau) \end{bmatrix}. \tag{9}$$

The stabilization problem is generally given by

$$\dot{\mathbf{x}}(\tau) = f(\tau, \mathbf{x}, \mathbf{u}), \tag{10}$$

where $\mathbf{u} = g(\tau, \mathbf{x})$ is the control law. Such a control law is usually called "static feedback", because is memoryless in respect to the state vector $\mathbf{x}$. Sometimes $\mathbf{u}(\tau - \tau_0)$ is a time-delayed control function, since the attempt of designing a state feedback control that depends on the measurement of a set of output variables often introduces a certain delay τ_0 on the response [11]. In that cases, it is common to design a dynamic feedback control $\mathbf{u} = g(\tau, \mathbf{x}, \mathbf{z})$, where $\mathbf{z}$ is the solution of a dynamical system driven by $\mathbf{x}$, given by $\dot{\mathbf{z}} = h(\tau, \mathbf{x}, \mathbf{z})$.

We are interested in a disturbance rejection problem, so we should design a feedback control law $\mathbf{u}$ such that the origin $\mathbf{x} = 0$ is an asymptotically stable equilibrium point of the closed-loop (10), regardless the fact that a more general solution could be adopted to stabilize the system in respect to an arbitrary steady-state point $\mathbf{x}_{ss}$. Since the system is linear, and assuming the possibility of making a continuos reading of the state variables at each time τ, the closed-loop system can be written in the form

$$\dot{\mathbf{x}}(\tau) = \mathbf{A}(\tau) \cdot \mathbf{x}(\tau) - K\mathbf{B} \cdot \mathbf{x}(\tau - \tau_0), \tag{11}$$

where the control law $\mathbf{u} = -K\mathbf{x}(\tau - \tau_0)$ preserves the linearity of the open-loop system. Here, $\mathbf{B}$ is a matrix of parameters. In the case of no delay, $\tau_0 = 0$, we know that the origin is an asymptotically stable point if, and only if, $\mathbf{A} - K\mathbf{B}$ is Hurwitz, i.e., if the real part of the ist eigenvalues are negative. For the general case $\tau_0 \neq 0$, we should use numerical calculations.

Although the parameters Δ and ϵ are time independent, it is expected that they contain a certain error, since they depend upon the experimental conditions, in a more realistic approach. Therefore, the robustness of the negative feedback control system presented in (11) should be tested. For the sake of illustration, in fig. (IV)we present several situation of stabilization for an unstable solution of (7). First, we set either the delay time τ_0 and the errors associated to the parameters to zero. Sencond, we assume the existence of a finite time-delay in the response of the control law function. Finally, we

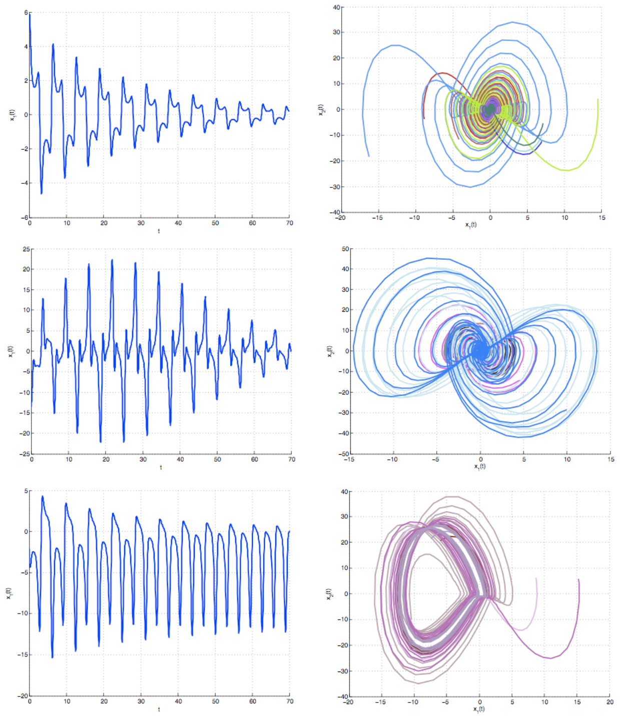

Fig. 4. (Color Online) Examples of control of the unbounded solution presented in fig. (3). Solutions obtained in the time domain (left) and in the phase-space domain (right) for five different orbits. From top to bottom: Simple negative feedback; negative feedback with time delay of $\tau_0 = 0.3$, and negative feedback with gaussian noise ($\sigma_\Delta = 2$, $\sigma_\epsilon = 2$).

assume that the errors on the parameters follow a gaussian distribution with zero mean value and variances σ_Δ and σ_ϵ.

In the case where $\tau_0 = 0$ and $\sigma_\Delta = \sigma_\epsilon = 0$, the unstable solution of the open-loop system is stabilized at the origin, without exhibiting any transient regime. In the second stabilization scheme, the time delay is set to 0.3, the solution tends to zero after a transient regime around $\tau_{trans} = 20$. In the last case, we can observe that the introduction of a bandwidth of $\sigma_\Delta = \sigma_\epsilon = 2$ in the parametric noise lead to a quasi-periodic stable solution, in opposition to the previous situations. However, because our aim is to avoid the occurrence of instabilities rather than zero-point stabilization, the later simulation suggests that this control system is robust to the noise.

We have restricted our discussion to the disturbance rejection problem for reasons that have to deal with the experimental conditions. In a typical MOT experiment, we are interested in cooling down the atoms, by decreasing their kinetic energy to zero. Instabilities obviously bring the system out of such configuration, preventing the atoms to be cooled and trapped. The design of a control system based on a defined steady-state configuration would also lead to similar results, since the disturbance rejection problem is only a particular case of the later.

V. Conclusion

The description of a cold Bose gas with a set of effective fluid equations opens place to the occurrence of plasma-like waves. Because of the trapping used in the typical experimental setups, the system oscillates at the dipole frequency, which is roughly defined by the magnetic field conditions. In such conditions, it is possible to predict the couple between the waves and the dipole oscillations. This system of coupled modes is described by a time-dependent model that generalizes the well-known Mathieu equation. It is shown that this equation exhibits unstable solutions, which may result in a source of instabilities that should be stabilized in the context of experimental work. Hence, the design of a negative feedback control system is of great interest. In the present work, we show that a simple time-dependent control law, based on the measurement of the open-loop state variables, should be considered as an efficient way of stabilizing the system. We purpose an extension of the usual techniques in control of linear time-dependent systems by including delaying and noise in the response functions.

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