Networks of Mixed Canonic-Dissipative Systems and Dynamic Hebbian Learning

Julio Rodriguez
École Polytechnique Fédérale de Lausanne
STI / IMT / LPM1
Lausanne 1015
Switzerland
Email: julio.rodriguez@epfl.ch

Max-Olivier Hongler
École Polytechnique Fédérale de Lausanne
STI / IMT / LPM1
Lausanne 1015
Switzerland
Email: max.hongler@epfl.ch

***Abstract*— We consider a collection $\{\mathcal{O}_k\}_{k=1}^N$ of interacting parametric *mixed canonical-dissipative systems*, (MCD). Each individual $\mathcal{O}_k$, exhibits, in absence of interaction, a limit cycle $\mathcal{L}_k$ on which the orbit circulation is parameterized by $\omega_k(t)$. The underlying network defining the interactions between the $\mathcal{O}_k$'s is assumed to possess a diffusive Laplacian matrix. For each $\mathcal{O}_k$, we construct a class of position- and velocity-dependent interactions which lead to a dynamic learning process of the Hebbian type (DHL). More precisely, the interactions affect the circulation parameterization $\omega_k(t)$ and the DHL mechanisms manifests itself by asymptotically driving the system towards a consensual (oscillatory) global state in which all $\mathcal{O}_k$ share a common circulation parameterization ω_c. It is remarkable that for our class of interactions, we are able to analytically calculate ω_c which, in our case, is independent of the topology of the connecting network. However, the coupling network topology explicitly controls the relaxation rate via the spectral gap of the underlying adjacency matrix (i.e. the so called *Fiedler number* of the associated graph). Finally, we report several numerical illustrations which enable to observe the DHL mechanisms at work and confirm our theoretical assertions.**

***Keywords*— mixed canonic-dissipative systems, limit cycles oscillators, dynamic Hebbian learning, consensual states, diffusive coupling, Laplacian matrix, algebraic connectivity.**

I. Introduction

In a recent paper [1], L. Righetti et al. show how to implement what they call a Dynamic Hebbian Learning (DHL) process by coupling nonlinear parametric oscillators with an external time-dependent signal. As a paradigmatic illustration, they consider an non-autonomous parametric Hopf oscillator (HO), defined, in its phase space, by the system of equations:

$$\text{HO}\begin{cases} \dot{x} &= +\omega y + \left(1 - x^2 - y^2\right) x + \epsilon \sin(\Omega t), \\ \dot{y} &= -\omega x + \left(1 - x^2 - y^2\right) y, \\ \dot{\omega} &= \epsilon \sin(\theta(t)) \sin(\Omega t), \end{cases} \tag{1}$$

where ϵ is a small positive constant, $\sin(\Omega t)$ externally perturbs the basic dynamics of the HO (Ω is a positive constant) and where $\theta(t) := \arctan(\frac{y(t)}{x(t)})$. The DHL process manifests itself by the fact that the circulation parameterization (i.e. here the basic frequency of the underlying HO) $\omega(t)$ does, asymptotically converge, to Ω, the frequency of the external input signal. In other words, the external signal "*plastically*" deforms the original limit cycle dynamics. We speak about plasticity to reflect the fact that, once this deformation is realized, it definitely subsists even if the external input is removed. This generic behavior can be qualitatively understood by the fact that the external perturbing signal gradually affects the circulation parameterization ω on the limit cycle $\mathcal{L}$, (for Eqs.(1), $\mathcal{L} := \left\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\right\}$), but leaves the shape of $\mathcal{L}$ essentially invariant.

The core of the present paper is to substitute in Eqs.(1) the role played by the external signal by the dynamics delivered by other limit cycle oscillators and then, to study the resulting mutual DHL process. More generally, we will consider a collection $\{\mathcal{O}_k\}_{k=1}^N$ of independent *mixed canonical-dissipative systems* (MCD) as introduced in [2] and [3], which exhibit limit cycles $\mathcal{L}_k$ and different individual $\omega_k(t)$, $k = 1, 2, \ldots, N$ on $\mathcal{L}_k$. The action of dissipative mechanism is to stabilize the orbits on $\mathcal{L}_k$ and the canonic part of the vector field (i.e. its Hamiltonian part) is responsible for the circulation on the limit cycles. In our class of models, the mutual interactions between the $\mathcal{O}_k$'s are characterized by:

a) *a network $\mathcal{N}$ of diffusively coupled $\mathcal{O}_k$*'s - i.e. the row elements of the associated Laplacian coupling matrix of the network add to zero.

b) *a dynamic Hebbian learning mechanism* (DHL). We allow the $\omega_k(t)$ to behave as additional variables and we implement couplings between these variables with the whole dynamics. Qualitatively speaking, the DHL coupling rule essentially affects the circulation parameterization on the limit cycles $\mathcal{L}_k$ while keeping the shape of $\mathcal{L}_k$ approximately unchanged.

The DHL process and the resulting "*plasticity*" of the dynamics confers a fundamentally different perspective compare to the yet abundantly studied synchronization networks of limit cycle oscillators. Indeed, interactions of the DHL type offer the possibility to drive the dynamics into a global (identical for all $\mathcal{O}_k$'s), stable oscillatory state which, once reached, remains "frozen" even when the interactions are removed. This final oscillatory behavior shared by all $\mathcal{O}_k$'s will be called the *consensual oscillatory state*. In this context, a (non-exhaustive) list of natural issues, to be addressed in this paper, will be:

1) How to calculate the circulation parameterization $\omega_c(t)$

characterizing the final consensual state ?

2) How does the consensual circulation parameterization depend on the Laplacian matrix associated to network ?

3) How does the network influence the convergence rate towards the the final consensual state?

In this contribution, we propose, in section II, the construction of an analytically soluble class of coupled oscillators with mutual interactions leading to a DHL rule. A paradigmatic illustration of this class of dynamics is thoroughly studied in section III where explicit and fully analytical answers to questions 1) to 3) can be given. Future research perspectives and conclusion will be found in section V.

II. Construction of a DHL dynamical network

The collection $\{\mathcal{O}_k\}_{k=1}^N$ of oscillators will be chosen to belong to the class of mixed canonical-dissipative systems which we briefly expose in II-A.

A. Mixed Canonic-Dissipative systems

A member of our collection $\{\mathcal{O}_k\}_{k=1}^N$ will be defined as:

$$\mathcal{O}_k \left\{ \begin{array}{lcl} \dot{x}_k &=& +\omega_k \frac{\partial H_k}{\partial y_k} \;+\; g_k(H_k)\frac{\partial H_k}{\partial x_k}, \\ \dot{y}_k &=& \underbrace{-\omega_k \frac{\partial H_k}{\partial x_k}}_{\text{conservative evolution}} \;+\; \underbrace{g_k(H_k)\frac{\partial H_k}{\partial y_k}}_{\text{dissipative evolution}}, \end{array} \right. \tag{2}$$

where $H_k : \mathbb{R}^2 \to \mathbb{R}^+$ and $g_k : \mathbb{R}^+ \to \mathbb{R}$. The H_k's functions are C^2 and positive definite and play the role of Hamiltonians (i.e. energy). In the sequel, we shall assume that $H_{k(x_k,y_k)} = \mathcal{E}_k$ uniquely defines a set of closed (concentric) curves $\mathcal{L}_{k(\mathcal{E}_k)}$ in $\mathbb{R}^2$ that surrounds the origin. The g_k's functions are C^1 and $g_k(H_{k(x_k,y_k)})$ are non-conservative terms which, according to the value of H_k, feeds or dissipates energy from the Hamiltonian system. In particular, if $g_k(H_{k(x_k,y_k)})$ vanishes for $H_{k(x_k,y_k)} = \mathcal{E}_k$, the dynamics is purely conservative (i.e. only the *canonical* part drives the dynamics) and we therefore have:

$$H_{k(x_k,y_k)} = \mathcal{E}_k \text{ defines the } \textit{limit cycle } \mathcal{L}_{k(\mathcal{E}_k)}$$

with

$$\mathcal{L}_{k(\mathcal{E}_k)} := \left\{(x,y) \in \mathbb{R}^2 | H_{k(x,y)} = \mathcal{E}_k\right\}.$$

The stability of the $\mathcal{L}_{k(\mathcal{E}_k)}$'s will be determined by:

$$\begin{array}{lcl} \begin{array}{l} g_k(H_k) > 0 \text{ in } \mathcal{A}_k \\ g_k(H_k) < 0 \text{ in } \mathbb{R}^2 \setminus \bar{\mathcal{A}}_k \end{array} & \Rightarrow & \mathcal{L}_{k(\mathcal{E}_k)} \text{ is stable,} \\ & & \\ \begin{array}{l} g_k(H_k) < 0 \text{ in } \mathcal{A}_k \\ g_k(H_k) > 0 \text{ in } \mathbb{R}^2 \setminus \bar{\mathcal{A}}_k \end{array} & \Rightarrow & \mathcal{L}_{k(\mathcal{E}_k)} \text{ is unstable,} \end{array} \tag{3}$$

where $\mathcal{A}_k$ stands for the interior of $\mathcal{L}_{k(\mathcal{E}_k)}$, (i.e. $\mathcal{A}_k := \{(x,y) \in \mathbb{R}^2 | H_{k(x,y)} < \mathcal{E}_k\}$). Therefore, for $g_k(\mathcal{E}_k) = 0$ and when $\mathcal{L}_{k(\mathcal{E}_k)}$ is stable, the energy-type control $g_k(H_{k(x_k,y_k)})$ drives all orbits towards the stable limit cycle $\mathcal{L}_{k(\mathcal{E}_k)}$ which is hence an attractor. The system defined by Eqs.(2) belongs to the general class of mixed canonical-dissipative dynamics (MCD) (c.f [2], [3] and [4]). In the sequel, we shall make use of the short hand notation:

$$P_k(x_k, y_k, \omega_k) := +\omega_k \frac{\partial H_k}{\partial y_k}(x_k,y_k) + g_k(H_{k(x_k,y_k)}) \frac{\partial H_k}{\partial x_k}(x_k,y_k),$$

$$Q_k(x_k, y_k, \omega_k) := -\omega_k \frac{\partial H_k}{\partial x_k}(x_k,y_k) + g_k(H_{k(x_k,y_k)}) \frac{\partial H_k}{\partial y_k}(x_k,y_k).$$

Observe that in Eqs.(2), we restrict our study to non-parametric MCD for which ω_k are constant.

Having defined the individual dynamics, it is now time to characterize the interactions.

B. Network of diffusively coupled oscillators

The interactions between the MCD's given by Eqs.(2) will be realized via a simply connected network $\mathcal{N}$ with N edges without loop (i.e. its adjacent matrix A is such that, for the j^{th} edge, $A_{j,j} = 0$, $j = 1, 2, \ldots, N$ and $A_{i,j} \in \{0, 1\}$ for $j \neq i$). Let L be the associated Laplacian matrix, namely $L = A - D$, where D is the diagonal matrix with $D_{j,j}$ being the degree of edge j. Accordingly, we now consider the dynamics:

$$\mathcal{O}_k \left\{ \begin{array}{lcl} \dot{x}_k &=& P_k(x_k, y_k, \omega_k) \;+\; C_k x \\ \dot{y}_k &=& \underbrace{Q_k(x_k, y_k, \omega_k)}_{\text{MCD}} \;+\; \underbrace{C_k y}_{\text{diffusive coupling}} \end{array} \right. \tag{4}$$

with $C_k x$ and $C_k y$ reading as:

$$C_k x := \epsilon_1(x,y) \sum_{j=1}^N L_{k,j} x_j \quad \text{and} \quad C_k y := \epsilon_2(x,y) \sum_{j=1}^N L_{k,j} y_j,$$

where $0 \leq \epsilon_l(x,y) < \epsilon$, $l = 1, 2$ not simultaneously vanishing and $x := (x_1, \ldots, x_N)$, $y := (y_1, \ldots, y_N)$.

Finally, we now introduce the DHL process into the dynamics.

C. Dynamic Hebbian learning for Mixed Canonic-Dissipative systems

Directly inspired from Eqs.(1), we now propose our generalized DHL in the context of Eqs.(4). The dynamical system is given by:

$$\mathcal{O}_k \left\{ \begin{array}{lcl} \dot{x}_k &=& P_k(x_k, y_k, \omega_k) + \; C_k x, \\ \dot{y}_k &=& Q_k(x_k, y_k, \omega_k) + \; C_k y, \\ \dot{\omega}_k &=& \underbrace{\mathsf{K}_k [Dy\; C_k x - Dx\; C_k y],}_{\text{DHL mechanism}} \end{array} \right. \tag{5}$$

where

$$Dy := \eta_1(x,y) \sum_{j=1}^N \frac{\partial H_j}{\partial y_j},$$

$$Dx := \eta_2(x,y) \sum_{j=1}^N \frac{\partial H_j}{\partial x_j}.$$

with $0 \leq \mathsf{K}_k \leq \kappa$ is a set of learning coupling strengths and $0 \leq \eta_l(x,y) \leq \eta$, $l = 1, 2$ are not simultaneously vanishing.

Observe at this point that the dynamics defined by Eqs.(5) exhibit the salient features of the basic model given by Eqs.(1). We namely have:

a) when $C_k x = C_k y = 0$ and for appropriate choices of the $g_k(H_{k(x_k,y_k)})$ terms, (see Eq.(3)), the dynamics exhibits a stable limit cycle $\mathcal{L}_k$,

b) on the limit cycle $\mathcal{L}_k$, the dynamics obeys a (conservative) canonical Hamiltonian motion,

c) a DHL type mechanism explicitly affects the circulation parameterization $\omega_k(t)$ of the orbits on $\mathcal{L}_k$.

For simplicity and without lost of generality, in what follows we shall systematically take $\epsilon_1(x,y) = \epsilon_2(x,y) = 1$ and $\eta_1(x,y) = \eta_2(x,y) = 1$ in Eqs.(5).

Proposition 1: Let $\mathsf{K}_k > 0$, for all k in the system defined by Eqs. (5). Then:

$$J := \sum_{k=1}^{N} \frac{\omega_k(t)}{\mathsf{K}_k} \tag{6}$$

is a constant of the motion.

Proof:

$$\begin{aligned}
\sum_{k=1}^{N} \frac{\dot{\omega}_k}{\mathsf{K}_k} &= \sum_{k=1}^{N} Dy\, C_k x - \sum_{k=1}^{N} Dx\, C_k y \\
&= Dy \sum_{k=1}^{N}\sum_{j=1}^{N} L_{kj} x_j - Dx \sum_{k=1}^{N}\sum_{j=1}^{N} L_{kj} y_j \\
&= Dy \sum_{j=1}^{N} x_j \sum_{k=1}^{N} L_{kj} - Dx \sum_{j=1}^{N} y_j \sum_{k=1}^{N} L_{kj} \\
&= 0.
\end{aligned}$$

where the last equality identically vanishes due to the diffusive character of the coupling matrix L.

■

Proposition 2: Assume that we have a collection of identical MCD systems (i.e. $H_k \equiv H$ for all k) admitting, in absence of coupling, the same stable limit cycle $\mathcal{L}_c := \mathcal{L}_{k(\mathcal{E}_c)}$ for all k (i.e. for a fixed energy level $\mathcal{E}_c$ common to all oscillators, we suppose that $g_k(\mathcal{E}_c)) = 0$ for all k). Then, the synchronized orbit given by $\mathcal{S}(t) := (x_s(t), y_s(t), \omega_c, \ldots, x_s(t), y_s(t), \omega_c) \in \mathbb{R}^{3N}$, with $\omega_c = \text{constant}$ and with:

$$g(H_{(x_s(t),y_s(t))}) = g\left(\mathcal{E}_c\right) = 0, \tag{7}$$

is an exact solution of the dynamical system defined by Eqs.(5).

Proof: For the synchronized orbit, we have $x_k(t) = x_s(t)$, $y_k(t) = y_s(t)$ and $\omega_k(t) = \omega_c(t)$ for all k. The diffusive nature of the coupling, implies that the terms $C_k x = C_k y = 0$ and therefore $\dot{\omega_k}(t) = 0$. Hence, the $\omega_c(t)$ are identically a constant written as ω_c. ■

So far, we have introduced a globally non-conservative dynamical system $\mathbb{R}^{3N}$ given by Eqs.(5) for which an explicit orbit $\mathcal{S}(t)$ is known. In addition, our dynamics possesses one constant of the motion J given by Eq.(6). One therefore may now question whether the orbit $\mathcal{S}(t)$ corresponds to a stable solution of the globally non-conservative dynamics. As usual, by linearizing the dynamics around $\mathcal{S}(t)$ produces information regarding its stability - this will be explicitly performed in section III for systems with an underlying circular symmetry. At this stage and to make head on, assume that $\mathcal{S}(t)$ is indeed a stable solution of the dynamics given by Eqs.(5) and that we have $\lim_{t\to\infty} \omega_k(t) = \omega_c$ for all k. Hence, ω_c corresponds to the **consensual circulation parameterization on the common limit cycle** $\mathcal{L}_c$. In this case, Proposition 1 and 2 provide explicit answers to questions 1) and 2) raised in the introduction. Indeed, Eq.(6) enables us to write:

$$\text{if} \quad \lim_{t\to\infty} \omega_k(t) = \omega_c \quad \text{then} \quad \sum_{k=1}^{N} \frac{\omega_c}{\mathsf{K}_k} = \sum_{k=1}^{N} \frac{\omega_k(0)}{\mathsf{K}_k}$$

and therefore, we end with:

$$\omega_c = \frac{\sum_{j=1}^{N} \frac{\omega_j(0)}{\mathsf{K}_j}}{\sum_{j=1}^{N} \frac{1}{\mathsf{K}_j}}. \tag{8}$$

From Eq.(8), we then conclude that the consensual circulation parameterization ω_c **depends on the distribution of initial conditions** $\{\omega_k\}$ **and on the coupling strength** K_k **for** $k = 1, 2..., N$ **but does not depend on the coupling matrix** L **and therefore not on the topology of the coupling network**. However, we shall see that L directly affects the convergence rate towards the consensual orbit $\mathcal{S}(t)$.

III. Network of coupled Hopf oscillators

In this section, we focus on the situation where $H_k \equiv H$ for all k and where the underlying Hamiltonian reads as $H(x,y) = H(x^2 + y^2) = H(r^2)$. The circular symmetry implies that the consensual limit cycle $\mathcal{L}_c$ is a circle and the circulation is a uniform rotation with the consensual frequency given by Eq.(8). Due to the cylindrical symmetry, it is advantageous to express the dynamics in polar coordinates:

$$\mathcal{O}_k = \left\{ \begin{array}{lcl}
\dot{r}_k &=& 2(1 - r_k^2) r_k + \sum_{j=1}^{N} L_{k,j} \cos(\phi_k - \phi_j)) \\
\dot{\phi}_k &=& -2\omega_k - \frac{1}{r_k} (\sum_{j=1}^{N} L_{k,j} r_j \sin(\phi_k - \phi_j)) \\
\dot{\omega}_k &=& \mathsf{K}_k \left[\sum_{l=1}^{N} (\sum_{j=1}^{N} L_{k,j} r_l r_j \sin(\phi_l - \phi_j)) \right].
\end{array} \right. \tag{9}$$

Note that in the non-parametric case (i.e. when $\omega_k(t) = \omega_k$), the phase dynamics in Eqs.(9) coincides with the Kuramoto model in presence of a general coupling network as discussed in [5]. Here, the exact solution of Eqs. (9) on which perturbations will now be added, simply reads as:

$$\begin{aligned}\mathcal{S}_{pc}(t) &= (r_s(t), \theta_s(t), \omega(t), \dots, r_s(t), \theta_s(t), \omega_s(t)) \\ &= (1, -2\omega_c t, \omega_c, \dots, 1, -2\omega_c t, \omega_c) \in \mathbb{R}^{3N}.\end{aligned} \tag{10}$$

Rearranging the variables in Eqs.(9) by using the permutation $3(k-1)+n \mapsto N(n-1)+k$ ($k = 1, \dots, N \quad n = 1, 2, 3$) and linearizing around $\mathcal{S}_{pc}(t)$ enables us to write:

$$\begin{pmatrix} \dot{\rho} \\ \dot{\delta} \\ \dot{\epsilon} \end{pmatrix} = \begin{pmatrix} L - 4\mathrm{Id} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & L & -2\mathrm{Id} \\ \mathbb{O} & -2\,[K]\,L & \mathbb{O} \end{pmatrix} \begin{pmatrix} \rho \\ \delta \\ \epsilon \end{pmatrix} \tag{11}$$

where Id is the identity matrix, $[K]$ is a diagonal matrix with $\mathsf{K}_1, \mathsf{K}_2, \dots, \mathsf{K}_N$ on the diagonal and where $\rho := (\rho_1, \dots, \rho_N)$, $\delta := (\delta_1, \dots, \delta_N)$ and $\epsilon := (\epsilon_1, \dots, \epsilon_N)$ are perturbations. To fulfill the conservation law given by Eq.(6), we further impose the that:

$$\sum_{j=1}^{N} \frac{\epsilon_j(0)}{\mathsf{K}_j} = 0, \quad (\text{here } \mathsf{K}_k \text{ is constant for all } k). \tag{12}$$

To explicitly exhibit the influence of the network, we focus on the case where $\mathsf{K}_k := \mathsf{K}$ for all k. Since L is symmetric, then there exists an orthogonal matrix V such that $V^\top L V$ is a diagonal matrix $[\lambda]$ with its spectrum $\{\lambda_k\}_{k=1}^N$ on the diagonal. The network being connected, there exists a unique j such that λ_j is zero and the rest of the spectrum are all strictly negative. Without lost of generality, we assume $\lambda_1 = 0$. Changing the basis of the system by means of a (3×3)-bloc matrix with $V^\top$ on its diagonal, gives us:

$$\begin{pmatrix} \dot{\tilde{\rho}} \\ \dot{\tilde{\delta}} \\ \dot{\tilde{\epsilon}} \end{pmatrix} = \begin{pmatrix} [\lambda] - 4\mathrm{Id} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & [\lambda] & -2\mathrm{Id} \\ \mathbb{O} & -2\mathsf{K}[\lambda] & \mathbb{O} \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{\delta} \\ \tilde{\epsilon} \end{pmatrix}. \tag{13}$$

The upper left $(N \times N)$-bloc in Eqs.(13) has N real negative eigenvalues and the rest of the system is described by the following (2×2)-blocs:

$$\begin{pmatrix} \dot{\tilde{\delta}}_k \\ \dot{\tilde{\epsilon}}_k \end{pmatrix} = \begin{pmatrix} \lambda_k & -2 \\ -2\mathsf{K}\lambda_k & 0 \end{pmatrix} \begin{pmatrix} \tilde{\delta}_k \\ \tilde{\epsilon}_k \end{pmatrix}. \tag{14}$$

For $k = 1$, we have:

$$\dot{\tilde{\delta}}_1 = -2\tilde{\epsilon}_1 \quad \dot{\tilde{\epsilon}}_1 = 0.$$

This is a direct consequence of the conservation law Eq.(6) and on the restriction imposed by Eq.(12). For $k \neq 1$, the eigenvalues of the System 14 are:

$$\alpha_\pm = \frac{1}{2}\lambda_k \pm \frac{1}{2}\sqrt{\lambda_k^2 + 16\mathsf{K}\lambda_k} < 0.$$

For a simple, connected graph, the spectrum $\{\lambda_k\}_{k=1}^N$ is negatively defined (c.f. [6]) which ensures an (exponential) asymptotic convergence to the consensual state. More precisely, we have a stable focus for $\lambda_k \in]-16\mathsf{K}, 0[$ and a stable node for $\lambda_k \in]-\infty, -16\mathsf{K}[$. The relaxation time is given by $\tau_{relax} = \mathcal{F}^{-1}$ where $\mathcal{F}$ is the algebraic connectivity (i.e. the *Fiedler number* - c.f. [7]) of the coupling network. Remember that $\mathcal{F}$ is the largest, non-vanishing, eigenvalue of the associated Laplacian matrix.

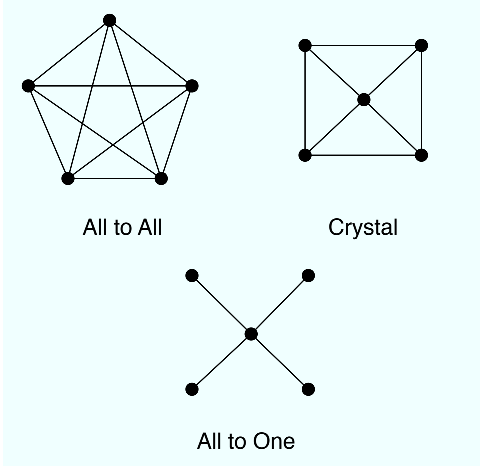

Fig. 1. Three types of network topologies with respective Fiedler number $\mathcal{F}_{(\cdot)}$: "All to All" ($\mathcal{F}_{\text{AtA}} = -5$), "Crystal" ($\mathcal{F}_{\text{Cry}} = -3$) and "All to One" ($\mathcal{F}_{\text{AtO}} = -1$).

IV. Numerical simulations

In Figures 2, 3 and 4, we report numerical simulations performed with five Hopf oscillators defined when $H(x, y) = x^2 + y^2$ and $g(H) = 1 - H$. Three different topologies of the interaction network are considered: "All to All", "Crystal"and "All to One" (c.f. Figure 1).

The learning mechanism can be observed in Figures 2, 3 and 4 and the final consensual frequency is given by Eq.(6). All three figures have the same time scale so that we can fully appreciate the fact that the convergence rates $\rho_{(\cdot)}$ clearly obey:

$$\mathcal{F}_{\text{AtA}} < \mathcal{F}_{\text{Cry}} < \mathcal{F}_{\text{AtO}} \Rightarrow \rho_{\text{AtA}} > \rho_{\text{Cry}} > \rho_{\text{AtO}}.$$

The smaller the Fiedler number, the faster the convergence and thus, the convergence rate does explicitly depend on the topology of the network.

V. Conclusions and perspectives

Among the numerous possibilities of implementing the DHL learning rule, networks of limit cycle oscillators with adapting frequencies offer a yet unexplored research topics with potential for applications. In this note, we are able to explicitly appreciate the interplay between the DHL learning rule on one hand and the connectivity of the underlying interaction network on the other hand. In particular, the possibility to analytically calculate the consensual circulation parameterization (c.f. Eq.(6)) characterizing the circulation of the final common attractor and the observation that the the topology of the network participate only to the convergence rate are truly remarkable features. At this preliminary stage, we do not yet offer a complete and mathematically rigorous treatment of the rich underlying dynamics.

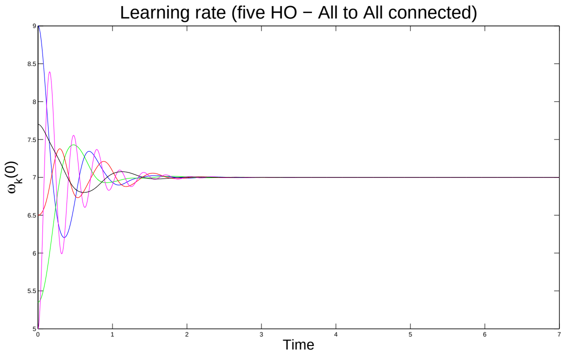

Fig. 2. Time evolution of the circulation parameterization for $\omega_k(t)$ for five Hopf oscillators with $\mathsf{K}_1 = 1$, $\mathsf{K}_2 = \frac{1}{2}$, $\mathsf{K}_3 = \frac{5}{4}$, $\mathsf{K}_4 = 5$, $\mathsf{K}_5 = \frac{1}{3}$ $\omega_1(0) = 9$, $\omega_2(0) = 5.35$, $\omega_3(0) = 6.5$, $\omega_4(0) = 5$, $\omega_5(0) = 7.7$ and with the network topology “All to All”. The consensual frequency, given by Eq.(8), is here $\omega_c = 7$.

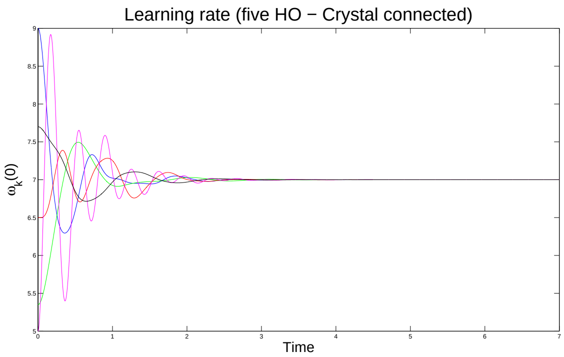

Fig. 3. Time evolution of the circulation parameterization for $\omega_k(t)$ for five Hopf oscillators with $\mathsf{K}_1 = 1$, $\mathsf{K}_2 = \frac{1}{2}$, $\mathsf{K}_3 = \frac{5}{4}$, $\mathsf{K}_4 = 5$, $\mathsf{K}_5 = \frac{1}{3}$ $\omega_1(0) = 9$, $\omega_2(0) = 5.35$, $\omega_3(0) = 6.5$, $\omega_4(0) = 5$, $\omega_5(0) = 7.7$ and with the network topology “Crystal”. The consensual frequency, given by Eq.(8), is here $\omega_c = 7$.

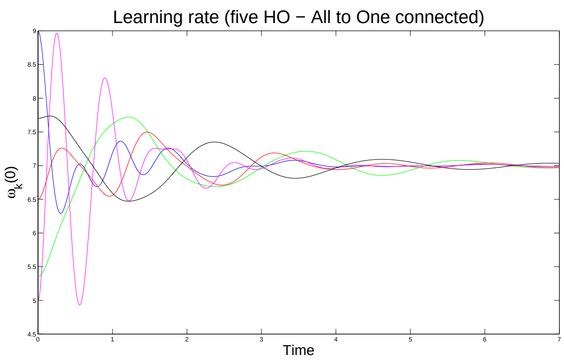

Fig. 4. Time evolution of the circulation parameterization for $\omega_k(t)$ for five Hopf oscillators with $\mathsf{K}_1 = 1$, $\mathsf{K}_2 = \frac{1}{2}$, $\mathsf{K}_3 = \frac{5}{4}$, $\mathsf{K}_4 = 5$, $\mathsf{K}_5 = \frac{1}{3}$ $\omega_1(0) = 9$, $\omega_2(0) = 5.35$, $\omega_3(0) = 6.5$, $\omega_4(0) = 5$, $\omega_5(0) = 7.7$ and with the network topology “All to One”. The consensual frequency, given by Eq.(8), is here $\omega_c = 7$.

Several open questions among which the characterization of the basin of attraction $\mathcal{B}$ of the consensual state, by constructing ad hoc Lyapunov functions, remain to be discussed. In particular, the dependence of $\mathcal{B}$ on the set of coupling parameters $\{\mathsf{K}_k\}_{k=1}^N$ and for coupling networks which can be modeled by multi-edge graphs remain yet to be unveiled.

Acknowledgment

The authors thank Ludovic Righetti for numerous and fruitful discussions shared during the elaboration of this work.

References

[1] L. Righetti, J. Buchli, and A. Ijspeert, “Dynamic hebbian learning in adaptive frequency oscillators,” *Physica D*, vol. 216, pp. 269–281, 2006.
[2] M.-O. Hongler and D. Ryter, “Hard mode stationary states generated by fluctuations,” *Zeitschrift fur Physik B*, vol. 31, pp. 333–337, 1978.
[3] F. Schweitzer, W. Ebeling, and B. Tilch, “Statistical mechanics of canonical-dissipative systems and applications to swarm dynamics,” *Physical Review E*, vol. 64, pp. 211 101–211 112, 2001.
[4] F. Schweitzer, *Brownian Agents and Active Particles*. Berlin, Germany: Springer, 2003.
[5] A. Jadbabaie, N. Motee, and M. Barahone, “On the stability of the kuramoto model of coupled nonlinear oscillators,” in *Proc. of the American Control Conference*, Boston, U.S.A., June 2004, pp. 4296–4301.
[6] J. L. Gross and J. Yellen, *Handbook of Graph Theory*. Florida, U.S.A.: CRC Press LLC, 2004.
[7] R. Olfati-Saber and R. M. Murray, “Consensual problems in networks of agents with switching topology and time-delays,” *IEEE Trans. Automat. Contr.*, vol. 49, pp. 1520–1533, Sept. 2004.

Julio Rodriguez is a Ph.D. Student at the Ecole Polytechnique Fédérale de Lausanne (EPFL), Switzerland. He received his Master in Mathematics in 2007 at Geneva University. After one year of teaching assistant at the Section of Mathematics, Geneva University, he joined the doctoral school of Manufacturing Systems and Robotics and started his Ph.D., centered on nonlinear dynamical systems and stochastic processes. In particular, his current research activities focus on assemblies of coupled nonlinear oscillators, synchronization and learning behaviors as well as collective motion of self-propelled agents.

Max-Olivier Hongler is Adjunct Professor at the Ecole Polytechnique Fédérale de Lausanne (EPFL), Switzerland. His basic education (at the Universities of Geneva, Switzerland, and Bristol, United Kingdom) is in theoretical physics, a domain in which he obtained a PhD degree from the University of Geneva in 1981 for his studies of non-linear stochastic models in off-equilibrium statistical physics. He then occupied research positions at the University of Texas, Austin, at the University of Toronto and at the University of Lisbon, Portugal. He was Invited Professor in theoretical physics at the University of Bielefeld, Germany, in 1992-1993. His main research interests include applied stochastic processes, optimal control and non-linear dynamics. At the EPFL, he teaches lectures in non-linear vibrations, in analytical dynamics, in supply-chains modeling and in stochastic models of manufacturing systems. He is Director of the Doctoral Program in Production and Robotics at the EPFL and Associate Editor of the IEEE Transactions on Industrial Informatics.