Recursive Backstepping for Synchronization of Chaotic Systems

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***Abstract*— A recursive technique based on the backstepping approach is introduced to build fast state observers for chaotic systems using the drive-response mechanism. A single scalar time series of the output is used to construct the synchronization algorithm that can perform as a nonlinear states observer as well. The proposed technique relies on using virtual states while employing control parameters that can adjust the convergence rate of the observed states. In addition, these control parameters can be used to improve the transient performance of the response system to accommodate small and large variations of the initial conditions, thus achieving superior performance to conventional synchronization techniques. The permissible range of the control parameters is estimated using Simple Lyapunov functions. Two benchmark chaotic systems are considered for illustration, namely the Lorenz and Chua systems. Tuning the control parameters and resolving the conflict between stability and agility are addressed while comparing a comparison with conventional synchronization methods. Implementation of the proposed technique in both analog and digital forms is also considered and experimental results are reported ensuring feasibility and real-time applicability.**

Keywords*— *Chaos Synchronization; Nonlinear Observers

I. INTRODUCTION

Since the early introduction of chaos synchronization [1,2], many techniques were reported in the literature with a wide variety of applications in different fields in science and engineering. Because of sensitivity to initial conditions, two trajectories emerging from two different closely initial conditions separate exponentially in the course of the time. As a result, chaotic systems defy synchronization [3]. Using a coupling or a forcing (periodical or noisy), two identical (or different) chaotic systems can adjust a given property of their motion to a common behavior. The most widely used types of synchronization, currently reported in the literature include complete synchronization [4], lag synchronization [5], generalized synchronization [6], phase synchronization [7], Q-S synchronization [8], and impulsive synchronization [9].

When performing complete synchronization of identical chaotic systems, a combination of drive and response systems is often utilized. The drive system represents the original chaotic system while the response system can represent a full or reduced states observer. In this type of synchronization both the drive and response systems have the same structure and dynamics except that the response system is driven by one of the signals of the drive systems. In this particular type of synchronization there is an interaction between one system and the other, but not vice versa, and synchronization can be achieved provided that all real parts of the Lyapunov exponents of the response system, under the influence of the driver, are negative [2]. Because both the drive and response systems follow the same chaotic trajectory over the course of time, the response system can be used to generate estimates of the immeasurable states of the drive system [10,11]. Therefore synchronization of dynamical systems and the design of state observers can be considered analogous [12] where many notations and terminologies are being shared between the two. In addition, Lyapunov functions play an important role in both synchronization of chaotic systems (originating from physics) and the design of states observers (originating from control engineering) as it is often used to prove the stability of the overall system as well as finding good estimates for the permissible range of the control parameters [13]. The drive-response synchronization scheme is essentially a control problem as the drive signal is used as a feedback signal for the response system such that the synchronization error is continuously attenuated. The error dynamics governing the difference between the driver and the response (observer) states are required to be globally stable while approaching zeros to ensure complete synchronization. This configuration found useful applications in both secure communication applications [14] and the construction of parameter identification algorithms [15].

When using the drive-response synchronization scheme the same set of parameters appears in both systems and consequently no control is done over the speed of convergence that is now strongly dependent on initial conditions. In this paper, a recursive procedure based on the backstepping approach is used such that, instead of observing the drive system states directly, virtual states are introduced in an intermediate stage that is characterized by having fast convergence rate via employing additional control parameters in either linear or nonlinear framework. This has the effect of speeding up the synchronization process resulting in better performance in applications such as secure communications.

Backstepping, a recursive technique, relies on introducing virtual reference models in designing model-based control systems and has proven to be very efficient for both regulation and tracking problems [16]. When using virtual reference models, it is possible to prescribe a target behavior for some or all of the system states and then use some of them as virtual controls to the output. This idea seems to be very appealing, especially when combined with Lyapunov-energy-like functions to design the control law for both linear and nonlinear systems [17]. In addition, backstepping designs are flexible and do not force the designed system to appear linear and can also avoid cancellation of useful nonlinearities and often introduce additional nonlinear terms to improve transient performance [18].

The need to develop faster states observers for chaotic systems for demanding applications such as secure communication and chaos control under the assumption that a single scalar time series, from the drive system, is available for measurement is the main motivation for the work presented in this paper. It is intended to use the drive-response synchronization method as a framework for the design of the fast states observer with the exception that both systems have non-identical structures. In addition, a feasible hardware-realizable implementation of the response system that can be easily tuned to achieve a fast synchronization with the drive system is introduced. This modified response system has an adjustable convergence rate, in contrast to the conventional synchronization mechanism in [2] where there are no control parameters to adjust the decay rate of the synchronization errors.

The rest of this paper is organized as follows: Sec. II introduces a detailed analysis of the design methodology by considering the Lorenz system as a chaotic benchmark model. Both simulations as well as experimental results are being reported to demonstrate the effectiveness of the proposed design. Extending the results to other chaotic systems is addressed in Sec. III where the Chua's model is considered. Comments regarding the practicality, advantages, and limitations of the proposed design are highlighted in Sec. IV as well as a brief comparison with other techniques reported in the literature and suggestions for future extensions.

II. THE LORENZ SYSTEM

The mathematical model of the Lorenz system, originally introduced in [19], is given by:

$$\begin{aligned} \dot{x}_1 &= -\sigma x_1 + \sigma x_2 \\ \dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= -\beta x_3 + x_1 x_2 \end{aligned} \tag{1}$$

where the nominal values of the parameters are 10.0, 28.0 and 8/3 corresponding to σ, ρ, and β respectively. These values are known to produce chaos for the free running case as illustrated in Fig. (1). Assuming that only x_1 is observable, the following two virtual states are now introduced:

$$f_i = x_i - k_{i1}x_1 - k_{i3}x_1^3 \,, i = 2,3 \tag{2}$$

where k_{21}, k_{23}, k_{31}, and k_{33} are design parameters. It is now required to use both f_2 and f_3 to recursively control the states x_2 and x_3 of the response system in order to continuously minimize the synchronization error while ensuring fast convergence rate. In addition, a deliberate cubic nonlinearity is introduced in Eq. (2) to increase the sensitivity of the virtual states to large synchronization errors; thus minimizing the effect of initial condition.

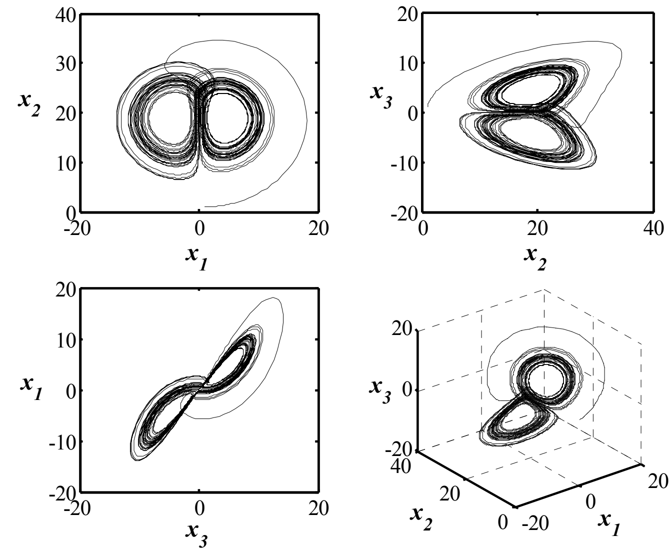


Figure 1. Chaotic behavior of the Lorenz system.

Differentiating Eq. (2), while using Eq. (1) for substituting for the states' derivatives, yields:

$$\begin{aligned} \dot{f}_2 &= [\rho x_1 - (f_2 + k_{21}x_1 + k_{23}x_1^3) - x_1(f_3 + k_{31}x_1 + k_{33}x_1^3)] \\ &\quad - (k_{21} + 3k_{23}x_1^2)[-\sigma x_1 + \sigma(f_2 + k_{21}x_1 + k_{23}x_1^3)] \\ &= -f_2 - x_1 f_3 - \sigma k_{21} f_2 - 3\sigma k_{23} x_1^2 f_2 + \varphi_2(x_1) \end{aligned} \tag{3}$$

where

$$\begin{aligned} \varphi_2(x_1) = &+x_1\left[\sigma k_{21}(k_{21}-1) + \rho - k_{21}\right] \\ &- x_1^2\left[k_{31}\right] \\ &+ x_1^3\left[k_{23}(\sigma k_{21}-1) + 3\sigma k_{23}(k_{21}-1)\right] \\ &- x_1^4\left[k_{33}\right] \\ &+ x_1^5[3\sigma k_{23}^2] \end{aligned} \tag{4}$$

and

$$\begin{aligned} \dot{f}_3 &= [-\beta(f_3 + k_{31}x_1 + k_{33}x_1^3) + x_1(f_2 + k_{21}x_1 + k_{23}x_1^3)] \\ &\quad - (k_{31} + 3k_{33}x_1^2)[-\sigma x_1 + \sigma(f_2 + k_{21}x_1 + k_{23}x_1^3)] \\ &= x_1 f_2 - \beta f_3 - \sigma k_{31} f_2 - 3\sigma k_{33} x_1^2 f_2 + \varphi_3(x_1) \end{aligned} \tag{5}$$

where

$$\begin{aligned} \varphi_2(x_1) = &-x_1\left[\sigma k_{31}(k_{21}-1) + \beta k_{31}\right] \\ &+ x_1^2\left[k_{21}\right] \\ &- x_1^3\left[3\sigma k_{33}(k_{21}-1) + \sigma k_{31}k_{23} + \beta k_{33}\right] \\ &+ x_1^4[k_{23}] \\ &- x_1^5[3\sigma k_{23}k_{33}] \end{aligned} \tag{6}$$

Introducing the following virtual errors:

$$e_i = f_i - \hat{f}_i \,, i = 2,3 \tag{7}$$

results in:

$$\begin{aligned} \dot{e}_2 &= -(1 + \sigma k_{21})e_2 - x_1 e_3 - 3\sigma k_{23} x_1^2 e_2 \\ \dot{e}_3 &= x_1 e_2 - \beta e_3 - \sigma k_{31} e_2 - 3\sigma k_{33} x_1^2 e_2 \end{aligned} \tag{8}$$

The following Lyapunov function is used to test for the global stability of the synchronization process:

$$L = 0.5(e_2^2 + e_3^2) \tag{9}$$

From which, we have:

$$\begin{aligned} \dot{L} &= e_2\dot{e}_2 + e_3\dot{e}_3 \\ &= -[(1+\sigma k_{21})e_2^2 + \sigma k_{31}e_2e_3 + \beta e_3^2] - 3\sigma k_{23}x_1^2e_2^2 - 3\sigma k_{33}x_1^2e_2e_3 \end{aligned} \tag{10}$$

which can be made negative definite via the following selection of the design parameters:

$$\begin{aligned} k_{33} &= 0 \\ k_{23} &\geq 0 \\ k_{21} &\geq \frac{\sigma k_{31}^2}{4\beta} - \frac{1}{\sigma} \end{aligned} \tag{11}$$

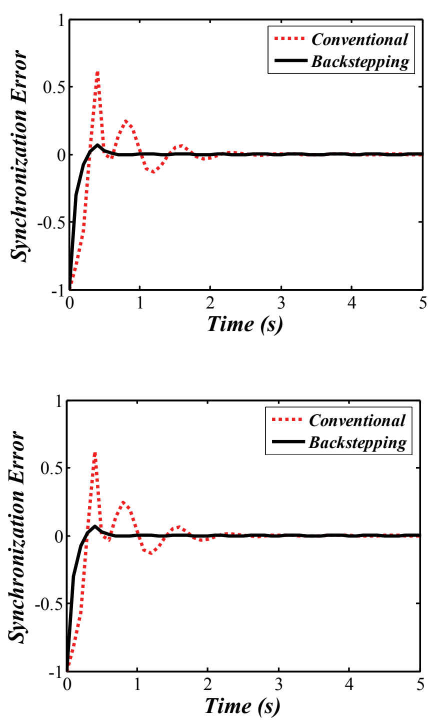
as Eq. (10) now reduces to:

$$\dot{L} \leq -(1+\sigma k_{21} - \frac{\sigma^2 k_{31}^2}{4\beta})e_2^2 - (\frac{\sigma k_{31}}{2\sqrt{\beta}}e_2 + \sqrt{\beta}e_3)^2 - 3\sigma k_{23}x_1^2e_2^2 \tag{12}$$

thus ensuring that as   the synchronization errors will decay to zero.

A. *Resolving the Conflect between Stability and Agility*

Equation (11) illustrates that a wide range of gains exist to satisfy the stability requirement. In addition, it is clear that the analysis of Eq. (10) can be greatly simplified by setting the terms corresponding to the cubic nonlinearity of Eq. (2) to zero; thus minimizing the tuning effort of the control parameters.

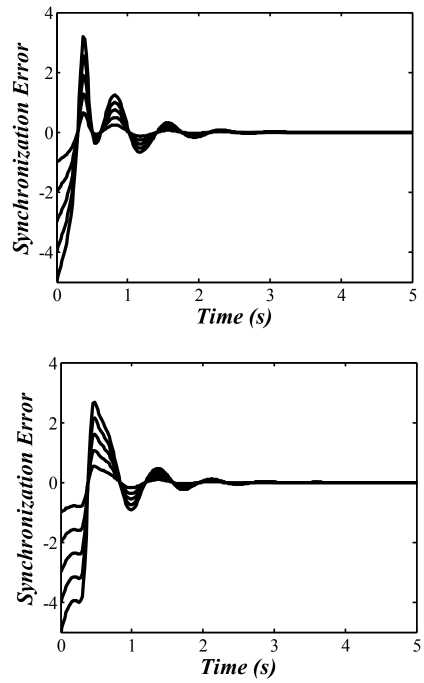


(a)

(b)

Figure 2. Synchronization errors for the two states x_1 in (a) and x_2 in (b).

When setting all the control parameters to zero, the designed observer reduces to the conventional drive-response synchronization method illustrated in [2] for which the dynamics for both systems are identical. In such case the synchronization speed is solely dependant on initial conditions.



(a)

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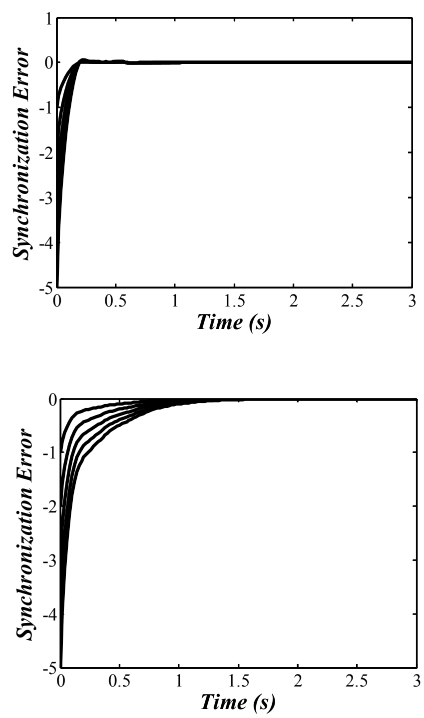
Figure 3. Initial conditions effect using conventional synchronization.



(a)

(b)

Figure 4. Initial conditions effect using backstepping synchronization.

Figure (2) illustrates the difference between the fast and smooth backstepping approach versus the slow and sluggish conventional approach where the gains for the backstepping case were adjusted to $k_{21} = k_{31} = 1.0$ and $k_{23} = k_{33} = 0.0$, while for the conventional case they were all set to zero. The effect of initial conditions on the transient performance of the synchronization process is shown in Figs. (3) and (4) illustrating the conventional and the backstepping methods respectively. The control parameters were all set to zero in Fig. (3) that clearly demonstrates the strong dependence for both x_2 and x_3 in (a) and (b) respectively on initial conditions. In contrast to the conventional synchronization method [2], Fig. (4) illustrates the superior transient performance for the backstepping case, where the control parameters were adjusted to $k_{21} = k_{31} = 1.0$, $k_{23} = 0.1$, and $k_{33} = 0.0$ for both x_2 and x_3 in (a) and (b) respectively.

B. Implementing the Proposed Technique

The proposed design, given by Eqs. (3-6), can be easily implemented using both analog and digital hardware. To verify this, a simplified version of the synchronization system is now illustrated for the case where only one design parameter is used, namely $k_{21} = 1$, while cancelling all other parameters.

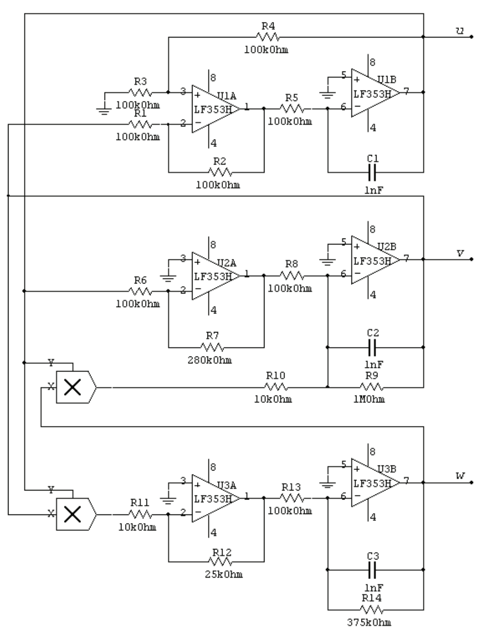


Figure 5. The drive circuit of the modified Lorenz system.

Figures (5) and (6) exemplifies one possible analog design, where the analog multipliers represent AD633AD, operational amplifiers are LF353, the supply was $V^+ = +15.0$, $V^- = -15.0$, and all resistors and capacitors have 1% tolerance. To meet the linearity constraints of the analog circuit, a linear transformation was applied such that the new states $u = 0.2\, x_1$, $v = 0.2\, x_2$, $w = 0.1\, x_3$, $g_2 = 0.2\, f_2$, and $g_3 = 0.1\, f_3$ were used instead of the original ones. Equation (13) illustrates the new system where a time scaling $\tau = 1$ ms was introduced to simplify the analog implementation.

$$
\begin{aligned}
\tau\dot{u} &= -\sigma u + \sigma v \\
\tau\dot{v} &= \rho u - v - 10uw \\
\tau\dot{w} &= 2.5uv - \beta w \\
\tau\dot{\hat{g}}_2 &= \rho u - \hat{g}_2 - 10u\hat{g}_3 - u - \sigma\hat{g}_2 \\
\tau\dot{\hat{g}}_3 &= 2.5u\hat{g}_2 - \beta\hat{g}_3 + 2.5u^2 \\
\hat{v} &= \hat{g}_2 + u \\
\hat{w} &= \hat{g}_3
\end{aligned}
\tag{13}
$$

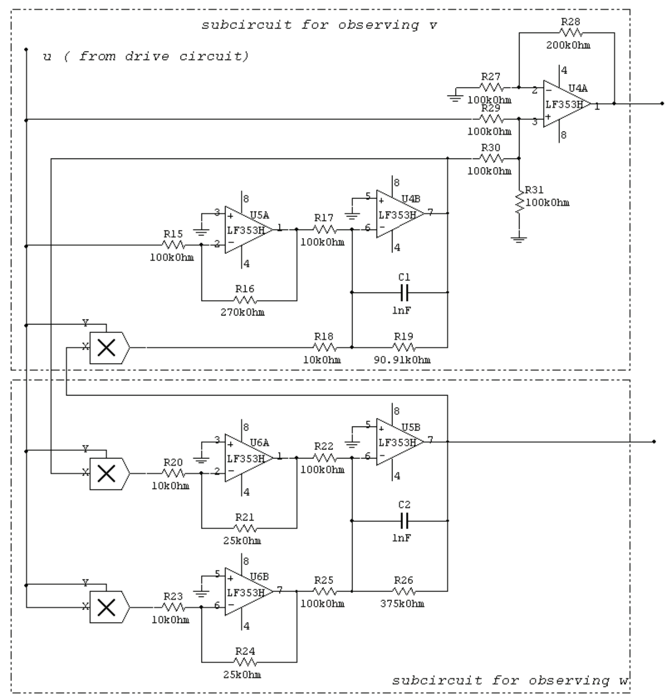


Figure 6. The response circuit of the modified Lorenz system.

The observed signals of the response circuit were measured using FLUKE 199C scopemeter and the analog data were converted to their digital equivalent and interfaced to the MATLAB environment using FLUKE SCC190. Figures (7) and (8) show the synchronization results for both v and w respectively reflecting the accurate convergence from the random initial conditions.

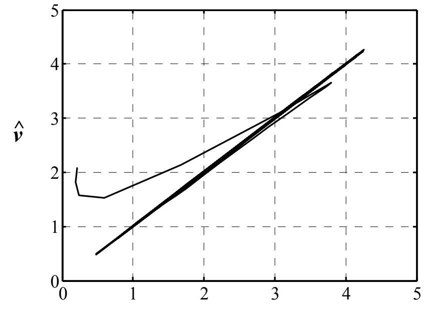


Figure 7. The synchronization performance for v.

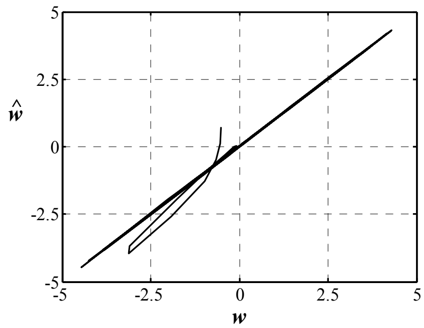


Figure 8. The synchronization performance for *w*.

III. THE CHUA'S SYSTEM

The next illustrative example is the Chua's system with smooth cubic nonlinearity for which the dynamical model is given in Eq. (14). This system is known to be a variant of the famous Chua's system with piecewise linear characteristics [20] that is easily implementable in both analog and digital hardware with typical applications in secure communications and synchronization-based systems.

$$\begin{aligned} \dot{x}_1 &= \alpha(x_2 - x_1^3 + cx_1) \\ \dot{x}_2 &= x_1 - x_2 + x_3 \\ \dot{x}_3 &= -\beta x_2 \end{aligned} \tag{14}$$

where the nominal values of the parameters are $\alpha = 10$, $\beta = 16$, and $c = 0.2$. Figure (9) illustrates the response of the system.

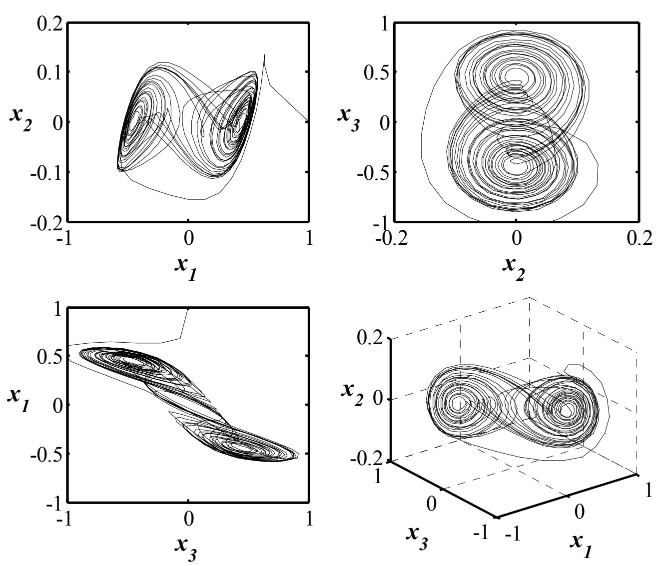


Figure 9. Chaotic behavior of the Chua's system.

Following the same methodology, outlined in the previous section, and assuming that only x_1 is available, the following virtual states can be introduced:

$$f_i = x_i - k_{i1}x_1 \,, i = 2,3 \tag{15}$$

from which we have:

$$\begin{aligned} \dot{f}_2 &= -f_2 + f_3 - \alpha k_{21} f_2 + \varphi_2(x_1) \\ \dot{f}_3 &= -\beta f_2 - \alpha k_{31} f_2 + \varphi_3(x_1) \end{aligned} \tag{16}$$

where

$$\begin{aligned} \varphi_2(x_1) &= x_1(1 - k_{21} + k_{31} - \alpha k_{21}^2 - \alpha c k_{21}) + \alpha k_{21} x_1^3 \\ \varphi_3(x_1) &= -x_1(\beta k_{21} + \alpha k_{21} k_{31} + \alpha c k_{31}) + \alpha k_{31} x_1^3 \end{aligned} \tag{17}$$

Introducing the following Lyapunov function:

$$L = 0.5(\mu e_2^2 + e_3^2) \,, \mu > 0 \tag{18}$$

results in:

$$\begin{aligned} \dot{L} &= \mu e_2 \dot{e}_2 + e_3 \dot{e}_3 \\ &= -\mu(1 + \alpha k_{21}) e_2^2 + e_2 e_3 (\mu - \beta - \alpha k_{31}) \end{aligned} \tag{19}$$

which can be made negative definite by the following choice of the control parameters:

$$\begin{aligned} k_{21} &\geq \frac{-1}{\alpha} \\ k_{31} &= \frac{\mu - \beta}{\alpha} \geq \frac{-\beta}{\alpha} \end{aligned} \tag{20}$$

Equation (20) verifies the flexibility of the proposed design as a wide range of the control parameters exist to resolve any conflicts between ensuring stability while maintaining fast response. In addition, Fig. (10) illustrates the improvement in the speed of convergence between the observed states (the response system) and their counterparts (the drive system). The backstepping gains were adjusted to $k_{21} = k_{31} = 1$, and the value of μ was set to 26, while for the conventional case they were set to zero

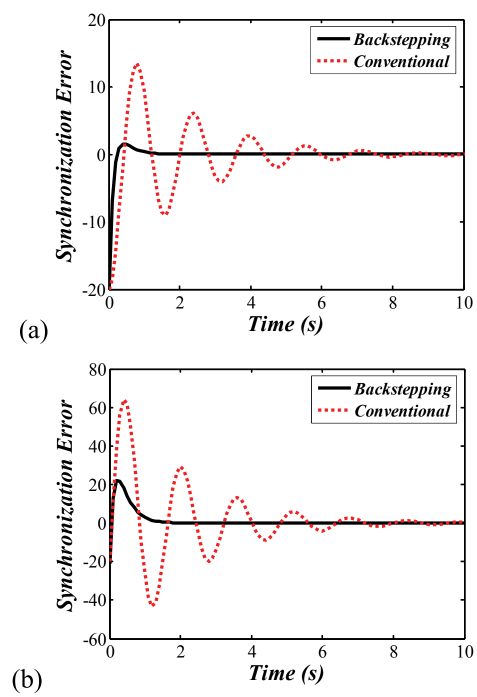


Figure 10. Synchronization errors for the two states x_1 in (a) and x_2 in (b).

IV. Discussion and Conclusion

The proposed technique was first exemplified by the Lorenz system and it was shown that the observed states do converge to their true values, thus completely synchronizing both the drive and response systems. The strategy of introducing virtual states to indirectly observe the true states of the drive system proved to have two advantages; first using one or more control parameters to adjust the speed of convergence, and second to have an overparameterized structure that allows introducing deliberate useful nonlinearities such that the transient performance of the synchronization errors is improved. The many possible implementations of the virtual system add both flexibility and versatility to the proposed design as depicted by the comparative analysis when the virtual system was reset to directly represent the response system. Tuning the control parameters requires the knowledge of the range for which the system representing the virtual synchronization errors is stable. This was seen to be an easy task via using a suitable Lyapunov function. In addition, it was further demonstrated that sensitivity to initial conditions was greatly reduced as the response system produced satisfactory results despite starting from different initial conditions in contrast to the conventional synchronization method where the results reflect large variance and strong dependence on initial conditions. This suggests that the proposed technique is well suited to applications in both chaos control and chaos synchronization, e.g. secure communications. Extending the application of the proposed technique to the Chua's system proved to be straight forward. For this particular case, and because of the existence of smooth cubic nonlinearity in the original drive system, there was no need to introduce more nonlinearities in the virtual model and only a simple linear model was proposed. This verifies the flexibility of the design and how it can be easily tailored to meet the structure of different systems and/or constraints imposed by the physical dynamic model. Also the choice of the suitable Lyapunov function can be a bottleneck in the design process as tuning the control parameters to achieve stable performance requires the proper choice of which output to be considered to construct the states observer, e.g. it was not possible to use x_3 to observe both x_1 and x_2 for the Lorenz system. Thus, which state is used for feedback and which states are to be observed, crucially affect the design process. Although the range of the control parameters is derived from the Lyapunov stability criteria, their maximum values should be constrained to ensure proper implementation of the design. This is due to the fact that overdriving the states observer can cause saturation in both the analog and digital implementations that can lead to instability as the observer dynamics are no longer valid.

Finally, the case when some or all of the drive system parameters are uncertain or unknown can be resolved by using a single control parameter during the implementation phase and consequently performing simple fine tuning for adjusting the gains. Another possible solution to this problem is to incorporate adaptive parameters' estimators; however, this will come at the expense of sacrificing the simplicity of the design and the increased order of the overall system.

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