System Identification using Symbolic Chaotic Sequence

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***Abstract*—In this paper, the problem of system identification using chaotic symbolic sequence is presented. We will consider the parameter estimation of linear moving average systems driven by chaotic sequences and formulate it as a semiblind identification scheme. The only knowledge that needs to be assumed at the estimator is about the dynamics of underlying chaotic system that generates the input sequence. To implement this estimator, we utilize expectation maximization (EM) algorithm. The sufficient statistics in the E-step is obtained with an unscented Kalman smoother (UKS). This intermediate step in system identification has similarity with chaos synchronization and hence we extend this idea to the synchronization of two chaotic systems under multipath. The estimation and synchronization performance of the proposed algorithm is evaluated using computer simulations.**

I. Introduction

Chaotic sequences have found many applications in secure communications, cryptography and digital watermarking due its wide–band nature. Synchronization of chaotic systems/maps is the backbone of many of these methods. The objective of any chaotic synchronization scheme is to get the trajectories of two systems with arbitrary initial conditions close to each other. Following the drive−response system suggested by Pecora and Carrol [1], various schemes have been proposed for the synchronization of chaotic systems [2]–[6]. One of the key issues in chaos synchronization is the presence of channel noise. Although many od these schemes can result in an acceptable level of synchronization, intermittent bursts of desynchronization is observed when channel noise is present [7]. A detailed account of the effect of the noise on synchronization is presented in [8]–[10]. One of the solutions suggested to synchronize chaotic systems when there is noise is the Kalman filter and its variants. Extended Kalman filter (EKF) has been shown to be successful in synchronizing chaotic systems/maps in stochastic environments. This ability of the EKF initiated a significant research interest [11], [12]. The EKF based scheme can be considered as a scheme which is capable of estimating the coupling strengths adaptively. Similarly, other variants of Kalman filters are also studied for chaotic synchronization hoping to improve the approximation errors introduced by EKF [13], [14].

Perturbation signal design for linear system identification is one of the key task [15]. For the persistent excitation, the input signal should have an impulse like auto correlation function. Filtered Gaussian noise is widely used as input sequence for system identification [16]. Random binary sequence, especially pseudo-random binary sequence (PRBS) are used as the input signal for system identification since it has desirable crest factor[1]. Chaos has been found to be an effective way to drive linear system for identification [17]–[19]. Chaotic time series is characterized by its noise like appearance and wide-band spectrum. Unlike white Gaussian noise, chaotic sequences are deterministic. It has been shown that the chaotic dynamics can act as a source of information and thus these sequences can be used as an alternative to the pseudo random noise signal. Chaotic sequences can be used either as an un–quantized sequence (i.e. the chaotic numeric sequences) or as chaotic symbolic sequences (Symbolic dynamics (SD) is the coarse-grain description of the chaotic dynamics and has been used for the analysis of chaotic systems/maps [20]). In any case, the potential of chaotic sequence for system identification still remain unexplored by the industry to a large extent. We believe it is due to the sensitive dependence of chaotic systems on its initial conditions and the difficulty in synchronizing noisy chaotic signals.

The method proposed in this paper attacks the problem of system identification and synchronization in a single framework. We formulate a semiblind system identification method which can be used for the synchronization of two chaotic systems. In system identification literature, semiblind estimation is used to emphasis that the estimator has only partial knowledge about the input signal. For example, in this case, only the dynamics of the chaotic that generates the input sequence is available at the estimator. We formulate the system identification problem as maximum likelihood estimation (MLE). Expectation maximization (EM) [21] is used for the recursive implementation of the MLE. Since the dynamics that generate perturbation signal is highly nonlinear, we use unscented Kalman smoother (UKS) for obtaining sufficient statistics in the expectation step (E-Step). This situation is very similar to the chaotic synchronization methods, where a estimate of the trajectory that generated the observation is estimated from the received signals. Thus, the problem is formulated as a joint estimation and synchronization problem, where we will estimate the channel coefficient and recover the

[1]Crest factor is the ratio of the peak value of a signal to its root mean square value. A lower crest factor represent more effective energy transfer to the system which results in enhanced signal to noise ratio.

transmitted chaotic trajectories recursively.

The current work is a continuation of the EM-EKS (extended Kalman smoother) based system identification presented in Ref. [22]. Though the proposed estimator is not restricted to any particular class of linear systems, we adopt a generic FIR system for simulation and performance comparison. For the second case, we run numerical simulations to study the estimation and synchronization performance. We have found that the performance of the proposed scheme is noticeable even at low SNR values. The rest of this paper is organized as follows. In section II, the problem of semi-blind identification is formulated for linear systems. We provide the numerical simulation results in section III. Finally, conclusions are drawn in section IV.

II. Problem Formulation

A. Parameter Estimation and Synchronization

We will consider a situation where a linear system excited by a chaotic sequence z_n and it could be a chaotic numerical sequence c_n or a chaotic symbolic sequence s_n. We observe the output of the system which is corrupted by noise. i.e.

$$y_n = \mathbf{h}^T \mathbf{z}_n + w_n. \tag{1}$$

where $\mathbf{z}_n = [z_n, \dots, z_{n-p+1}]$ is the input to the system with $\mathbf{h} = [h_1, \dots, h_p]$ and w_n is zero mean additive white Gaussian noise with variance σ_w^2. The sequence $c_n, n = 1, \dots, N$ with N as the total length of the input sequence is obtained by

$$c_{n+1} = f(c_n, \eta) \tag{2}$$

where $f(c_n, \eta)$ is a chaotic function parameterized on η. The symbolic sequence s_n is generated by

$$s_n = g(c_n) = \begin{cases} +1 & \text{if } \ c_n \geq \eta \\ -1 & \text{otherwise} \end{cases} \tag{3}$$

We need to estimate $\mathbf{h}$ by using only the observations and knowledge about the chaotic generator. Since we assume that the dynamics of the sequence generator is available, this method falls under semiblind identification. In our formulation, if the initial condition of the chaotic generator, c_0, is known the entire symbolic sequence can be reconstructed. This step has similarities with chaotic synchronization where the trajectories of one system are forced to follow the other. i.e the mean square of the error

$$e_n = c_n - \hat{c}_n, \tag{4}$$

should be minimal($\hat{c}_n$ is the estimated chaotic sequence).

The estimation problem becomes estimating $\mathbf{h}$ and $\sigma_{\mathbf{w}}^{\mathbf{2}}$ in addition to the initial condition. Let $\boldsymbol{\theta} = \{c_0, \sigma_w^2, \mathbf{h}\}$ are the set of parameters we need to estimate. This procedure can be treated as a batch approach for joint parameter estimation and synchronization.

B. The Proposed EM-UKS Estimator

In this section we formulate EM-UKS algorithm for system identification. EM is a standard tool for iterative maximum likelihood estimation [23]. It is a two stage algorithm which involves an expectation step (E–Step) and a maximization step (M-Step) in each iteration. After randomly initializing the unknown parameters, the algorithm performs E–Step using the current parameter estimates and M–Step by maximizing the expectation. In the next subsection, these two steps are explained.

1) E-Step: In the EM algorithm, missing variables are introduced as a part of the estimation process. In certain situation this variable is introduces as an artifact to make the problem tractable. In many other situations, this missing variable will be a part of the estimation problem [23]. In our problem, the state variables of the chaotic system are unobserved and it will naturally become the missing variable. We formulate an augmented state space model for our application. The three unknowns ($\boldsymbol{\theta}$) we try to estimate here is $\mathbf{h}, \sigma_w^2$ and c_0. We form the state vector $\mathbf{z}_n = [z_n, z_{n-1}, \dots, z_{n-(p-1)}]$ which is the missing variable and has the dynamics

$$\mathbf{z}_{n+1} = \begin{bmatrix} f(z_n) \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & 0 \\ \mathbf{I}_{p-1} & \mathbf{0} \end{bmatrix} \mathbf{z}_n \tag{5}$$

where $\mathbf{I}_{p-1}$ is an identity matrix of order $p-1$. In the E–Step we construct the complete statistics, $\mathcal{P}(\mathbf{Z}, \mathbf{y})$,with hidden variable $\mathbf{Z} = [\mathbf{z}_1, \dots, \mathbf{z}_N]$ which is an estimate of c_n and observances $\mathbf{y} = [y_1, \dots, y_N]$. Then the probability density function is given by

$$\mathcal{P}(\mathbf{Z}, \mathbf{y}) = \mathcal{P}(\mathbf{z}_1) \prod_{n=2}^{N} \mathcal{P}(y_n | \mathbf{z}_n), \tag{6}$$

since due to the deterministic nature, $\mathcal{P}(\mathbf{z}_{n+1} | \mathbf{z}_n) = 1$. We have

$$\begin{aligned} \mathcal{P}(\mathbf{z}_1) &= \mathcal{N}(0, \sigma_c^2 \mathbf{I}) \\ \mathcal{P}(y_n | \mathbf{z}_n) &= \mathcal{N}(\mathbf{h}^T \mathbf{z}_n, \sigma_w^2) \end{aligned} \tag{7}$$

Once this is obtained, the next step is to get the expectation Q as

$$\begin{aligned} Q &= \mathbb{E}\left[\mathcal{L}(\mathbf{Z}, \mathbf{y}))|\mathbf{y}, \boldsymbol{\theta}\right] \\ &= -\frac{1}{2}\ln(2\pi\sigma_c^2) - \frac{N}{2}\ln(2\pi\sigma_w^2) - \frac{1}{2\sigma_c^2}\mathbb{E}\left[\mathbf{x}_1^2|\mathbf{y}, \boldsymbol{\theta}\right] \\ &\quad - \sum_{n=1}^{N} y_n^2 - 2\sum_{n=1}^{N} y_n \mathbf{h}^T \mathbb{E}\left[\mathbf{z}_n|\mathbf{y}, \boldsymbol{\theta}\right] \\ &\quad + \sum_{n=1}^{N} \mathbf{h}^T \mathbb{E}\left[\mathbf{z}_n \mathbf{z}_n^T|\mathbf{y}, \boldsymbol{\theta}\right] \mathbf{h}. \end{aligned} \tag{8}$$

In order to compute the above expectation, we need to find the individual expectations

$$\begin{aligned} \mathbf{x}_n^s &= \mathbb{E}[\mathbf{z}_n|\mathbf{y}, \boldsymbol{\theta}] \\ \mathbf{P}_n^s &= \mathbb{E}[\mathbf{z}_n \mathbf{z}_n^T|\mathbf{y}, \boldsymbol{\theta}] \end{aligned} \tag{9}$$

A smoother can give these individual expectation values. Since the sequence generated from a nonlinear system, Unscented Kalman Smother (UKS) is used for its advantage over the traditional EKS [24]. We use UKS based on the Rauch–Tung–Striebel smoothing [25].

2) M-Step: In the M-Step, we use the result from the previous steps to do the maximization using the following steps.

$$\begin{aligned}
\hat{\mathbf{h}} &= \left[\sum_{n=1}^{N} \mathbf{P}_n^s\right]^{-1} \sum_{n=1}^{N} y_n \mathbf{x}_n^s \\
\hat{\sigma}_w^2 &= \frac{1}{N}\sum_{n=1}^{N}(y_n - \hat{\mathbf{h}}^T \mathbf{x}_n^s) \\
\hat{\sigma}_c &= c_0^s
\end{aligned} \tag{10}$$

These results are used in the next E–Step. This process is repeated until the iterations converge. The schematic of the proposed method is shown in figure 1.

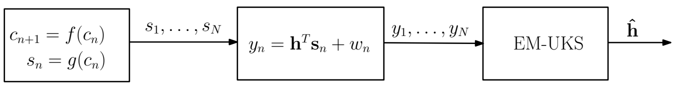

Fig. 1. Parameter estimation of a linear system using EM-UKS.

One of the motivations for this work is the interplay between the E–step and M–step. In the E–step, it is assumed that the estimated parameters are true and is used to obtain a smooth estimate of the state of the chaotic map from the observations. This is essentially similar to synchronization method where, system trajectories are estimated from observations. Once the smoothed states are obtained, this will be used for the parameter estimation. Thus, the overall procedure results in synchronized trajectories once the algorithm is converged.

III. Results and Discussion

In this section, we will discuss the result of the numerical simulations. The chaotic map we used for all the simulation is given by

$$c_{n+1} = \frac{\gamma c_n (1 - c_n^2)}{1 + \rho c_n^2} \tag{11}$$

where $\gamma = 5$ and $\rho = 2$. The chaotic attractor of the map is shown in Fig. 2. We set the threshold value $\eta = 0$. We analyze the parameter estimation as well as synchronization qualities of the proposed method.

A. Performance of the Proposed System on Parameter Estimation

For the analysis, we define the estimation error in the linear channel $\mathbf{h}$, $\mathrm{MSE}_{\mathbf{h}}$ as

$$\mathrm{MSE}_{\mathbf{h}} = \frac{1}{N}\sum_{i=1}^{N} \frac{||\mathbf{h} - \hat{\mathbf{h}}||}{P} \tag{12}$$

where N is the number of iteration carried out in each SNR values[2]. We compare this with both filtered Gaussian noise and PRBS based non–blind identification systems.

[2]Here, the SNR is defined as the power of the signal after threshold operation divided the power of the noise. This is identical to the typical formulation communication systems.

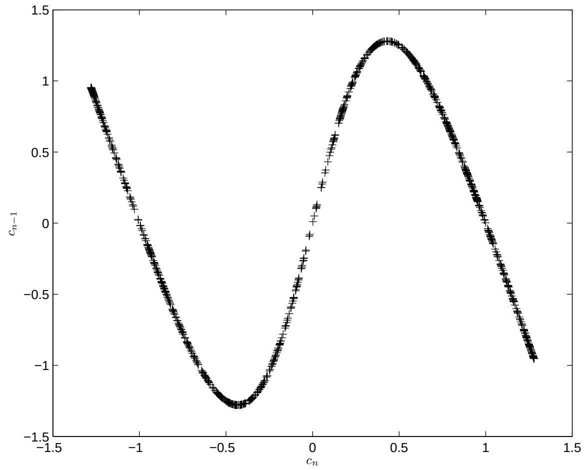

Fig. 2. Original Attractor.

We consider the system identification problem in a general settings with FIR coefficients $\mathbf{h} = [1, 0.6, 0.3]$. We run numerical simulations to study the effectiveness of the proposed scheme. $\mathrm{MSE}_{\mathbf{h}}$ is calculated according to Eq. (12) with $N = 16$. The result is plotted in figure 3. For comparison purpose, we have also plotted the non–blind identification schemes based on the white Gaussian noise and PRBS. Clearly, chaotic symbolic sequence based estimation scheme closely follows the training sequence based non–blind identification scheme at all SNR values. Compared to the chaotic numeric sequence, it is clearly an advantage. We will see the reason for this in the next subsection when we study the synchronization performance.

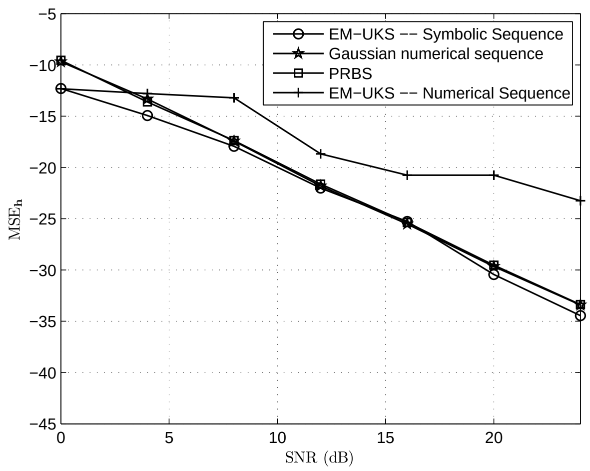

Fig. 3. SNR *vs* $\mathrm{MSE}_{\mathbf{h}}$.

As we know from general statical signal processing, it is desirable to have a large number of observations to improve the accuracy of the estimate. Next, we will study the effect of the number of sample N on the MSE performance. Figure 4 shows relationship between the length of the observations and $\mathrm{MSE}_{\mathbf{h}}$. We change the value of N from 16 to 512 by keeping channel noise at 20dB. For both chaotic symbolic and numerical sequences, the dependence of MSE on N is obvious. The chaotic symbolic sequence outperforms the numerical sequence in all the SNR values. Also, the rate of decrease in MSE is more prominent in the case of the symbolic sequence based identification. Since the increase in N increase the computational complexity, this information can be used for designing the receiver.

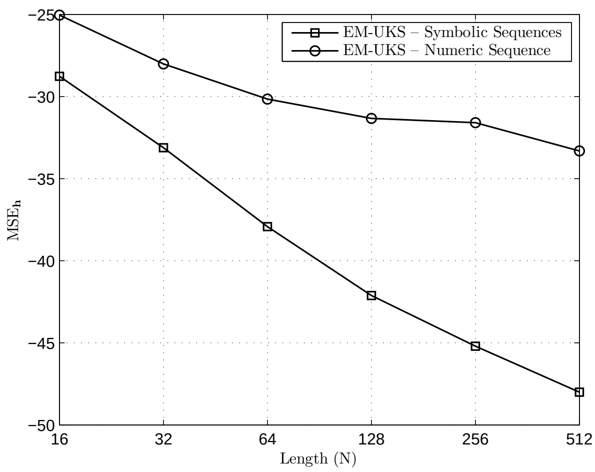

Fig. 4. Length *vs* $\mathrm{MSE}_{\mathbf{h}}$.

As an iterative algorithm, computational power requirement for EM algorithm is very high. It is desirable to achieve convergence with minimum iterations. To study effect of noise on the convergence of the EM algorithm, we consider two SNR values and performed a number simulations. The number of iterations required for convergence is plotted as histogram in figures 5 and 6. It can be seen that there is a strong dependency on the convergence of EM-UKS and noise. For chaotic symbolic sequences, at 20dB most of the iterations converge within seven or eight iterations while at 10dB EM-UKS takes at least 8 iterations. We observe few simulations takes close to 30 iterations to converge. Similar observations are made on the chaotic numeric sequence; however, compared to chaotic symbolic sequence it takes more number of iterations to converge. From all these studies, we can see that chaotic symbolic sequence based system identification scheme has very attractive performance measures despite the heavy computational load.

B. Performance of the Proposed System on Chaotic Synchronization

In this section we present the results of the proposed technique for the synchronization of chaotic sequences. We will start the analysis with un–quantized chaotic sequence. We consider a situation where, strong multipath and channel noise exists. We use the same channel as in the previous section for these simulations. The state space of the received signal (when the channel noise is 10dB) is given figure 7 and the corresponding reconstructed state–space is shown in figure 8. Clearly, even at 10dB noise, the attractor of the chaotic system

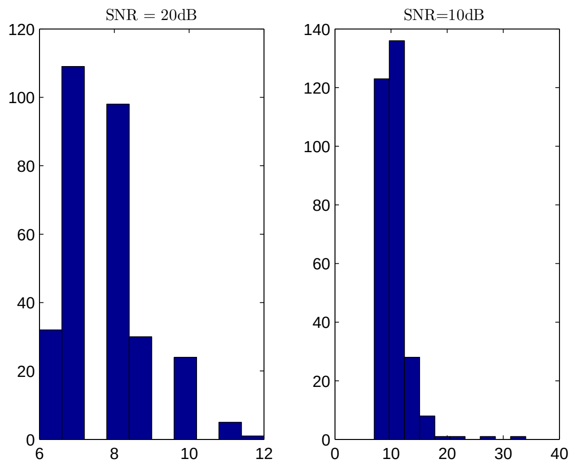

Fig. 5. Histogram of number of iterations taken for convergence: (a) 20dB and (b) 10dB.

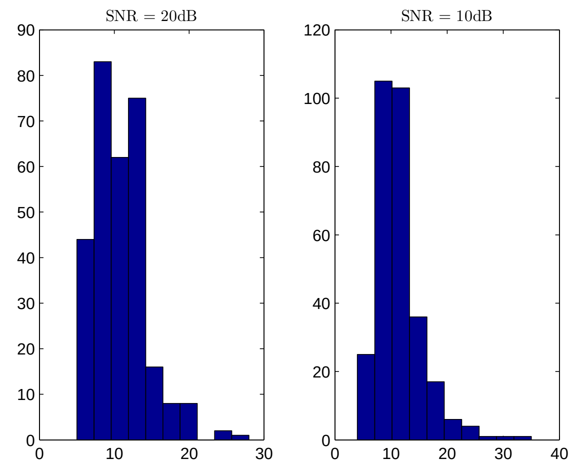

Fig. 6. Histogram of number of iterations taken for convergence: (a) 20dB and (b) 10dB.

is reconstructed very closely to the original one. For the higher SNR, qualitatively, the reconstructed attractor matches closely with the original one (figure 9). The original, received and reconstructed waveforms, when the channel noise is 10dB are shown in figure 10. We have observed no transient in our simulations which implies a quick synchronization.

Next, we will study when symbolic sequences are used for the transmission. From figure 11, we can see that the estimated and original sequence does not follow one to one. This is due to the approximation we used in the execration step. i.e. instead of finding the exact trajectory, its coarse representation estimated. Figure 12 shows the original and estimated symbolic sequences. The estimated symbolic sequence closely follows the corresponding symbolic sequence entering the channel. This method can be used with other symbolic sequence based synchronization schemes [26] in order to estimate the corresponding numeric sequence.

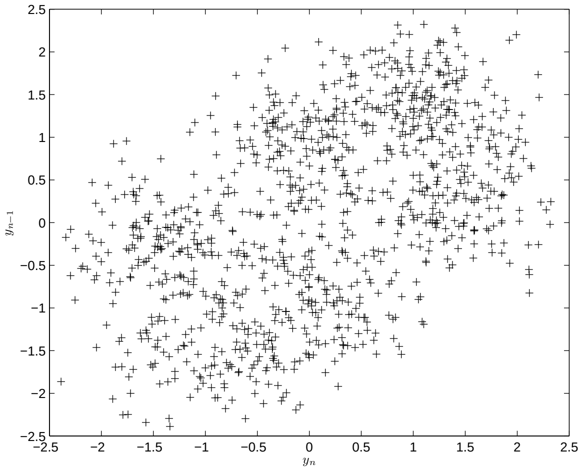

Fig. 7. State–space of the received signal when the channel noise is 10dB.

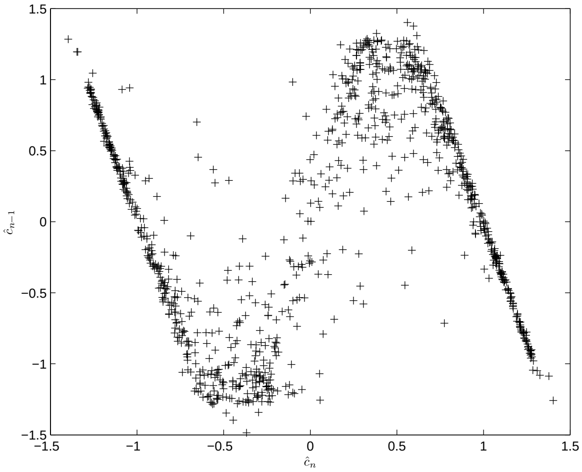

Fig. 8. Recovered state–space when the channel noise is 10dB.

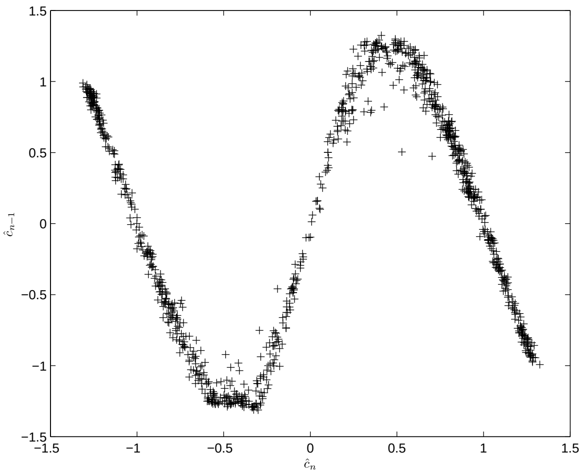

Fig. 9. Recovered state–space when the channel noise is 20dB.

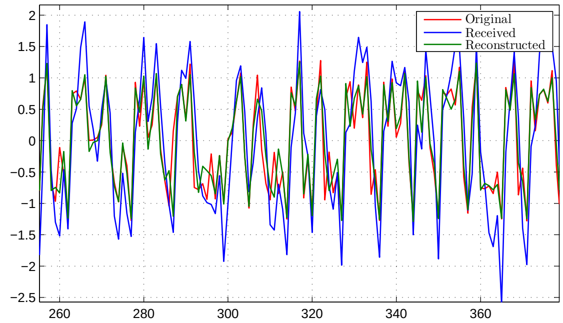

Fig. 10. Portion of the original, received, and recovered signals: Chaotic numeric sequence with channel noise 10dB.

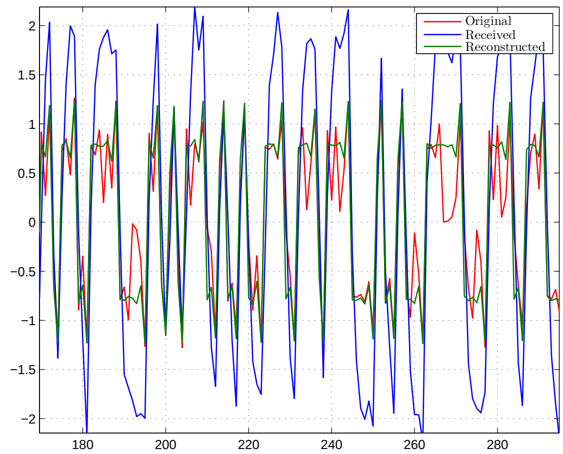

Fig. 11. Portion of the original, transmitted, and recovered signals: Chaotic symbolic sequence with channel noise 10dB.

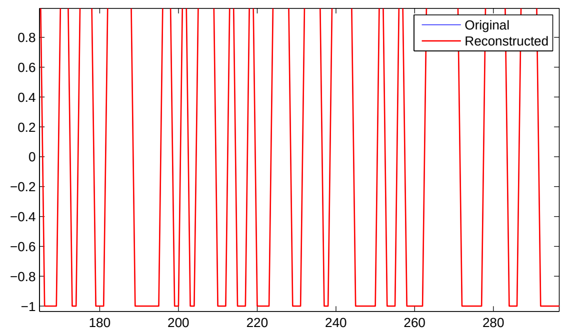

Fig. 12. Original and estimated symbolic sequence with channel noise 10dB.

IV. Conclusion

In this paper, the problem of chaotic synchronization is formulated as joint estimation and synchronization. By combining EM with UKS we form EM–UKS estimator for system identification. With this, the E–step acts as synchronization step and M–step acts as parameter estimation step. We compared the estimation and synchronization performance using numerical simulations. We found that the estimation performance of the proposed system is close to training based Gaussian white noise and PRBS in all SNRs when symbolic sequence is used. Similarly, at low and high SNRs, the proposed scheme was able to synchronize well.

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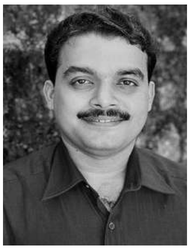

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